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Fishery Statistics Training

MATHEMATICS FOR FISHERY STATISTICIANS

by

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PREFACE

The purpose of this manual is to cover those topics of mathematics required by fishery statisticians and workers in other related fields, in order that they may have better understanding of statistics.

In writing this manual we have been influenced by our experience in teaching fishery statistics at various levels, i.e. elementary, intermediate and advanced in a variety of training centres in the world organized on a national or regional basis.

The manual is flexibly organized so that it can be used by researchers who have different levels of mathematical background. It is divided into three main parts:

PART I : Algebra
PART II : Calculus
PART III : Trigonometry

Each part has been arranged in such a way as to permit a wide selection of topics.

It is hoped that the users of the manual will acquire the essential ideas of mathematics needed for a better understanding of fishery statistics and sampling 1/.

1/ See also: Bazigos, G.P. (1974) - The Design of Fisheries Statistical Surveys - Inland Waters, FAO-FIPS/T133, p.89-100;
- Applied Fishery Statistics, FAO-FIPS/T135, p.39-42, p.147-152

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PART 1 - ALGEBRA

1.1 Introduction

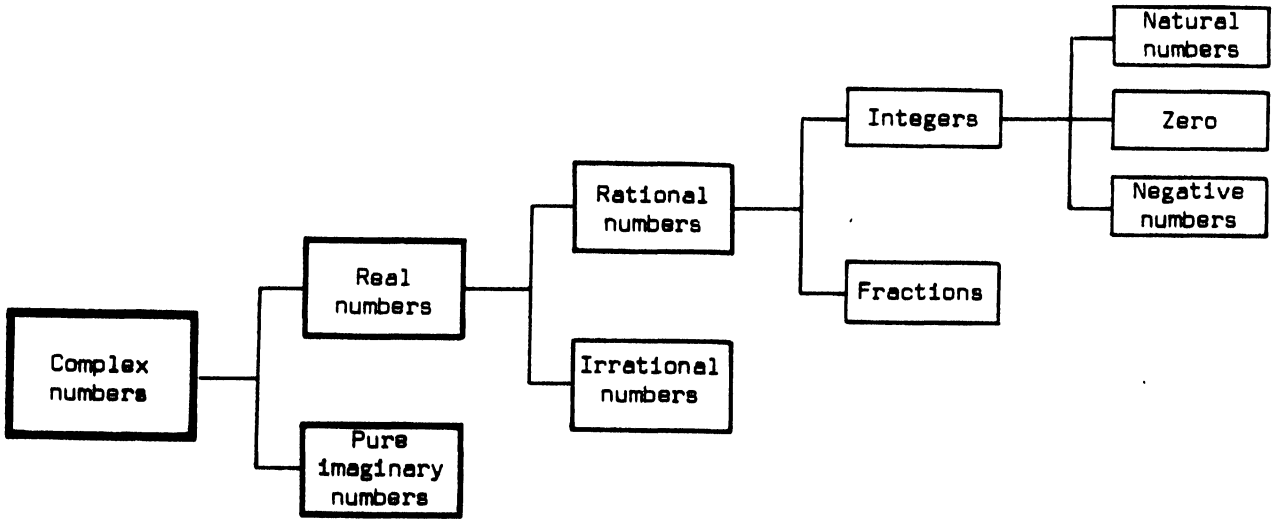
As a young child you first learnt to count, and thus became acquainted with the "natural numbers" 1/ 1,2,3,4,5 ... In your early study of arithmetic, you learnt how to add, subtract, multiply, and divide pairs of natural numbers. Although some divisions such as: $8 \div 4 = 2$ were possible, it soon became clear that new numbers had to be invented so as to give meaning to expressions like $8 \div 3$ and $5 \div 7$. To handle such situations, fractions were introduced, and the arithmetic of fractions was developed.

Later on, the system of "rational numbers" was developed consisting of the positive and negative integers, zero, and the positive and negative fractions.

When you were introduced to "irrational numbers" such as $\sqrt{2}$ and π, you were told that these could not be expressed as ordinary fractions.

The collection of the "rational" plus the "irrational" numbers is called the system of "real numbers".

The chart below shows the various types of numbers which have been invented and their relationships:



1/ Natural numbers are also called positive integers

In the summary below, we state the properties of the arithmetic of real numbers. The letters a,b,c stand for arbitrary real numbers:

Addition

$a + b$ is a unique real number	(Closure law)
$(a + b) + c = a + (b + c)$	(Associative law)
$a + 0 = 0 + a = a$	(Identity law)
$a + (-a) = (-a) + a = 0$	(Inverse law)
$a + b = b + a$	(Commutative law)

Multiplication

$a \times b$ is a unique real number	(Closure law)
$(a \times b) \times c = a \times (b \times c)$	(Associative law)
$a \times 1 = 1 \times a = a$	(Identity law)
$a \times \frac{1}{a} = \frac{1}{a} \times a = 1$ for $a \neq 0$	(Inverse law)
$a \times b = b \times a$	(Commutative law)

Distributive law

$$a \times (b + c) = (a \times b) + (a \times c)$$

1.2 Algebraic expressions

Certain symbols of operation, such as: $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, are common in Arithmetic and Algebra. To generate algebraic expressions we use letters which represent numbers in a general way and without any reference to quantities such as length, weight, etc. The following are examples of such expressions:

$$a + b, \quad 6a + \frac{3}{b^2}, \quad \frac{6x + 5x^2}{2 - 3x}, \quad \sqrt{3x^2 + 5y^2}$$

In other words, an algebraic expression is a combination of symbols which stand for numbers and for operations with them.

When the expression contains the symbols of operation + or -, those parts of the expression which they separate are called "terms". Specifically, the following terminology is used:

"Monomial" or "Term" is a combination of letters [1/] which does not contain either of the signs + or -, e.g. a, $3ab$, $\frac{6ab^2}{7}$, etc., are monomials.

Polynomial is the sum of a finite number of terms or monomials (a term may also include a negative sign), e.g. $1 - a$, $3x^2 + 5y$, $3b^2 - 5a^2 + 3a$, etc., are polynomials.

In writing algebraic expressions we employ "brackets" in order to indicate the order in which the operations are to be carried out. Thus,

$$5\,(x + y) - Z$$

means that we find the sum of x and y, multiply this by 5, and then subtract by Z. Similarly,

$$(a + b) \times (c + d)$$

means that we find the sum of a and b, and also of c and d, and then multiply the two results.

The expression 5b denotes a multiple of b and the number 5, which indicates the multiple, is called the coefficient of b. The coefficient may be a definite number like 3, called a numerical coefficient, or it may be a letter representing a number. For example, in the expression cx, c may be regarded as the coefficient of x [2/].

Like terms: In an algebraic expression, terms which involve the same letter(s) and differ only in the coefficients of this letter, are called like terms:

$$3a + 5b - a + 3b$$

In the above expression,

3a and a are like terms
5b and 3b are like terms

1/ See also Section 1.6

2/ If x is regarded as the coefficient of c, we would write the expression as xc

1.3 Elementary operations in Algebra

1.3.1 Addition and Subtraction of like terms

In algebra the operation of "finding the sum" is used to include both addition and subtraction. This is called the "algebraic sum".

The following relations are valid in algebraic summation:

$$(+a) + (+b) = a + b$$
$$(+a) - (+b) = a - b$$
$$(-a) + (-b) = -a - b$$
$$(-a) + (+b) = -a + b$$
$$(+a) + (-b) = a - b$$
$$(+a) - (-b) = a + b$$
$$(-a) - (-b) = -a + b$$
$$(-a) - (+b) = -a - b$$

The rule for adding together like terms is to add the coefficients of the terms (remember: $x = 1x$).

Examples:
$$a - a = 0$$
$$6a + 4a + a = 11a$$
$$9a - 3a + a = 7a$$
$$6x - 3y + 5x - 2y = (+6x + 5x) - (+3y + 2y) = 11x - 5y$$
$$3z + 6w - 2z - 5w = (+3z - 2z) + (+6w - 5w) = z + w$$

Evaluation by substitution:

In order to find the numerical value of an algebraic expression for definite numerical values of the letters composing it, we employ the following procedure:

(a) We simplify the expression by adding like numbers

(b) We substitute the numerical values for the letters

Example 1 : Find the numerical value of $3x - 5y + 4x + 7y - 2$, when $x = 2$ and $y = 3$:

(a) $(3x + 4x) + (7y - 5y) - 2 = 7x + 2y - 2$

(b) $7 \times (2) + 2 \times (3) - 2 = 18$

Example 2 : Find the numerical value of $3xy + 2yz - xy - \frac{z}{3} + 10$, when $x = 3$, $y = 2$, $z = 6$

(a) $(3xy - xy) + 2yz - \frac{z}{3} + 10 = 2xy + 2yz - \frac{z}{3} + 10$

(b) $2 \times (3 \times 2) + 2 \times (2 \times 6) - \frac{6}{3} + 10 = 44$

1.3.2 Multiplication

The following relations are valid in algebraic multiplication:

$(+a) \times (+b) = +ab$

$(+a) \times (-b) = -ab$

$(-a) \times (+b) = -ab$

$(-a) \times (-b) = +ab$

In Algebra the product of a number of factors is independent of the order in which they are multiplied:

$a \times b \times c = c \times a \times b = b \times c \times a$, etc.

Examples: $2a \times 3b = 6ab$

$5x \times 4z \times 2w = 40xzw$

$\frac{x}{3} \times 3y \times z = xyz$

1.3.3 Division

The following relations are valid in algebraic division:

$(+a) \div (+b) = +\frac{a}{b}$

$(+a) \div (-b) = -\frac{a}{b}$

$(-a) \div (+b) = -\frac{a}{b}$

$(-a) \div (-b) = +\frac{a}{b}$

1.4 Powers

The product of equal numbers is called a power[1/]. Thus,

$6 \times 6 = 6^2$ is called the second power of 6

$6 \times 6 \times 6 = 6^3$ is called the third power of 6

$6 \times 6 \times 6 \times 6 = 6^4$ is called the fourth power of 6 , etc.

Similarly,

$a \times a = a^2$ is called the second power of a

$a \times a \times a = a^3$ is called the third power of a

$a \times a \times a \times a = a^4$ is called the fourth power of a

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\underbrace{a \times a \times \ldots \times a}_{n} = a^n$ is called the n^{th} power of a

1.4.1 Multiplication of powers

Suppose we require to multiply two powers of a,

$$a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^{2+3} = a^5$$

In general,

$$a^m \times a^n = a^{m+n}$$

When two powers of the same number are multiplied, the index of the product is the sum of the indices of the factors.

Examples:

$$x^3 \times x^2 = x^{3+2} = x^5$$

$$2x^3 \times x^5 = 2x^{3+5} = 2x^8$$

$$\tfrac{3}{2}b^2 \times 6b^4 = 9b^{2+4} = 9b^6$$

$$a^3b \times a^2b^3 = a^3 \times a^2 \times b \times b^3 = a^{3+2} \times b^{1+3} = a^5b^4$$

1/ $a^4 = a \times a \times a \times a$; a^4 is the fourth power of a. The figure used in this way is called an index or exponent; it indicates the number of factors.

Power of a product:

By definition,

$$(ab)^2 = (ab) \times (ab) = a \times a \times b \times b = a^2b^2$$

When taking the power of a product, the index of the power is said to be distributed and applied to each factor of the product.

Examples: $(3xy)^2 = (3xy)\,(3xy) = (3 \times 3 \times x \times x \times y \times y)$

$$= 3^2x^2y^2 = 9x^2y^2$$

$$(y^2)^3 = (y^2) \times (y^2) \times (y^2) = (y \times y) \times (y \times y) \times (y \times y)$$

$$= y \times y \times y \times y \times y \times y = y^6$$

In general,

$$(x^m)^n = x^{mn}$$

$\therefore$ $(3a^2b^3)^2 = 3^2a^4b^6 = 9a^4b^6$

$\therefore$ $(\frac{1}{2}x^3y)^3 = \frac{1}{2^3}x^9y^3 = \frac{x^9y^3}{8}$

$\therefore$ Find the value of the expression $(3a^2x^3)^2$, when $a = 2$, $x = 1$:

$(3a^2x^3)^2 = 9a^4x^6 = 9 \times (2)^4 \times (1)^6 = 144$

1.4.2 Division of powers

Suppose that the power of a given number is divided by the power of the same number:

$$a^3 \div a^2 = \frac{a \times a \times a}{a \times a} = a^{3-2} = a$$

In general, $a^n \div a^m = a^{n-m}$. When dividing the power of a number by another power of the same number, subtract the index of the divisor from the index of the dividend.

Examples:

1) Divide $72a^4$ by $12a^2$,

$$\frac{72a^4}{12a^2} = 6a^{4-2} = 6a^2$$

2) Divide $6x^3$ by $3x^4$,

$$\frac{6x^3}{3x^4} = 2x^{3-4} = 2x^{-1} = \frac{2}{x}$$

3) Divide $3ax^4$ by $6a^2x^3$,

$$\frac{3ax^4}{6a^2x^3} = \frac{1}{2} \times a^{1-2} \times x^{4-3} = \frac{1}{2}a^{-1}x = \frac{x}{2a}$$

Properties of powers (summary):

1) $a^n \times a^m = a^{n+m}$

2) $\frac{a^n}{a^m} = a^{n-m}$, if $n > m$ (symbol > reads greater than)

3) $\frac{a^n}{a^m} = \frac{1}{a^{m-n}}$, if $n < m$ (symbol < reads less than)

4) $\frac{a^n}{a^m} = 1$, if $n = m$

5) $(a^n)^m = a^{mn}$

6) $(a \times b \times c \times \dots)^n = a^n \times b^n \times c^n \times \dots$

7) $(\frac{a}{b})^n = \frac{a^n}{b^n}$

1.4.3 Fractional exponents

Let c be a positive real number, and suppose that $b^n = c$. We would then say that "b is an n^{th} root of c". In symbols, we have:

$$b^n = c \quad \text{and} \quad b = \sqrt[n]{c} = c^{1/n}$$

Properties of powers with fractional exponents:

1) $a^{1/q} = \sqrt[q]{a}$

2) $a^{p/q} = \sqrt[q]{a^p}$

3) $(a^{p/q})^q = a^{pq/q} = a^p$

4) $\left((a^{1/q})^q\right)^p = a^p$

5) $a^{v/q} \times a^{u/q} = a^{v+u/q}$

6) $a^{v/q} \div a^{u/q} = a^{v-u/q}$, etc.

Note that, $\sqrt{a^2} = (a^2)^{1/2} = a^{2/2}$

Examples:

1) $\sqrt{a^2b^4} = (a^2b^4)^{1/2} = a^{2/2}b^{4/2} = ab^2$

2) $\dfrac{\sqrt{x^2y}}{\sqrt{x^3y^2}} = \dfrac{(x^2y)^{1/2}}{(x^3y^2)^{1/2}} = \dfrac{xy^{1/2}}{x^{3/2}y} = (xy^{1/2}) \div (x^{3/2}y)$

$= x^{1-3/2}y^{1/2-1} = x^{-1/2}y^{-1/2} = \dfrac{1}{x^{1/2}y^{1/2}} = \dfrac{1}{\sqrt{xy}}$

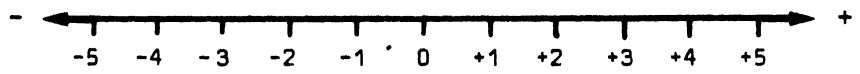
3) $x(-x)^{n-1} = x\left((-1)(x)\right)^{n-1} = x\left((-1)^{n-1}x^{n-1}\right)$

$(-1)^{n-1}xx^{n-1} = (-1)^{n-1}x^{n-1+1} = (-1)^{n-1}x^n$

1.5 Geometric presentation of real numbers

In the connection between arithmetic and geometry, the presentation of real numbers as points on a scale is most important:

Linear scale:



The important fact of this presentation is that every point corresponds to one and only one real number, and that every real number corresponds to one and only one point.

Properties:

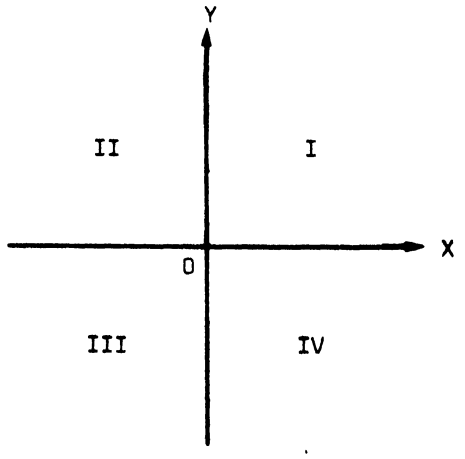
1) We say that a is greater than b (written: $a > b$) if a-b is positive. Similarly, a is less than b (written: $a < b$) if a-b is negative.

2) We can define a correspondence between the real numbers and the points of a directed line.

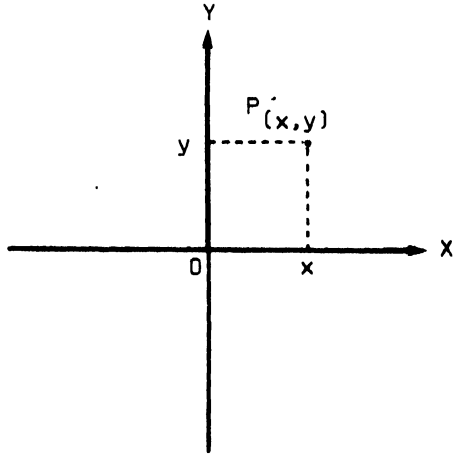
3) If a and b are the end points of a segment on a directed line (a,b), the length of the segment is the real number (b-a).

The use of real numbers in the plane:

We construct two perpendicular lines in the plane which we call the X-axis and the Y-axis:



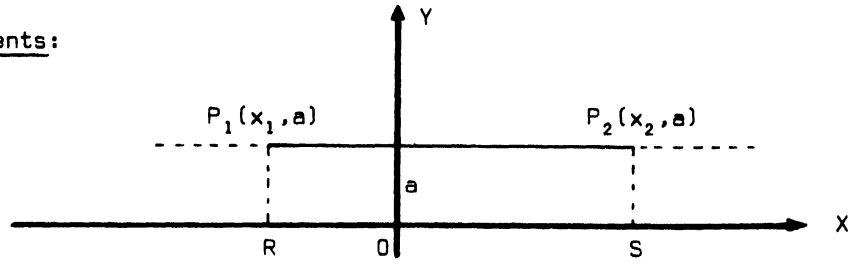
(a) The four quadrants of the plane



(b) The co-ordinates of the point P(x,y)

As indicated in the above graph (a), the lines divide the plane into four regions called "quadrants" which are numbered I, II, III, IV. Further, graph (b) shows that we can associate an ordered pair of real numbers (x,y) with each point P of the plane.

Lengths of segments:



Suppose P_1 and P_2 lie on a line parallel to the X-axis. We wish to have the expression of the length of P_1P_2. Draw P_1R and P_2S perpendicular to the X-axis. Then R has co-ordinates (x_1,o) and S has co-ordinates (x_2,o). Moreover, the lengths of P_1P_2 and RS are equal, since the opposite sides of a rectangle are equal. The length $RS = |x_2 - x_1|$. Hence, $P_1P_2 = |x_2 - x_1|$.

1.6 Polynomials

It has been discussed that polynomials can be expressed as a finite sum of unlike terms, each of which is either a number or consists of a numerical coefficient multiplied by the product of a finite number of distinct variables, of which each carries an exponent which is a positive integer:

$$3x^2 - x + \tfrac{1}{2}, \quad x^3 + \frac{1}{x} + 3, \quad \sqrt{2x} + 4x^2$$

1.6.1 Addition of polynomials

The process of addition can be conveniently carried out by arranging the work in columns, each column containing like terms.

Example 1 : Add $(3x^2 - 5y^2 + 3y + 2) + (x^2 - 2y + 5)$

The above example can be written:

$$\begin{array}{r} 3x^2 - 5y^2 + 3y + 2 \\ x^2 \qquad - 2y + 5 \quad (+) \\ \hline 4x^2 - 5y^2 + y + 7 \end{array}$$

Example 2 : Add $(5x^2y + x - 3xy^2 + 2) + (-2x + 3y + 7xy^2 - 5)$

The above example can be written:

$$\begin{array}{r} 5x^2y + x - 3xy^2 + 2 \\ -2x + 7xy^2 - 5 + 3y \quad (+) \\ \hline 5x^2y - x + 4xy^2 - 3 + 3y \end{array}$$

In order to subtract two polynomials, we convert the problem to addition and then proceed as above:

$$(4x^2 - 3xy + 2) - (5x^2 + x - 3)$$

is written as the sum,

$$(4x^2 - 3xy + 2) + (-5x^2 - x + 3) ,$$

$$\begin{array}{l} 4x^2 - 3xy + 2 \\ -5x^2 \qquad + 3 - x \qquad (+) \\ \hline - x^2 - 3xy + 5 - x \end{array}$$

or,

$$- x^2 - 3xy - x + 5$$

1.6.2 Multiplication of polynomials

The procedure for the multiplication of polynomials is based upon the method of multiplying monomials together with the repeated use of the distributive law. This reduces the problem of multiplying polynomials to that of multiplying monomials.

Example 1 : Multiply $(4a^2 + 5b + 1) \times (a^2 + 2b)$

The work can be conveniently arranged as shown below:

$$\begin{array}{l} 4a^2 + 5b + 1 \\ a^2 + 2b \qquad (\times) \\ \hline 4a^4 + 5a^2b + a^2 \\ \quad + 8a^2b \qquad + 10b^2 + 2b \\ \hline 4a^4 + 13a^2b + a^2 + 10b^2 + 2b \end{array}$$

Example 2 : Multiply

$$\begin{array}{l} 5x^4 - 8x^3 + x^2 + 5x - 3 \\ x^2 + 2x - 1 \qquad (\times) \\ \hline 5x^6 - 8x^5 + x^4 + 5x^3 - 3x^2 \\ \quad + 10x^5 - 16x^4 + 2x^3 + 10x^2 - 6x \\ \qquad - 5x^4 + 8x^3 - x^2 - 5x + 3 \\ \hline 5x^6 + 2x^5 - 20x^4 + 15x^3 + 6x^2 - 11x + 3 \end{array}$$

Binomial theorem:

By direct multiplication, as in the last section, we can easily calculate the following formulas:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The coefficients of these products have a pattern which is illustrated by the following scheme, known as Pascal's triangle:

$$(a + b)^0 = 1$$

$$(a + b)^1 = 1 \quad 1$$

$$(a + b)^2 = 1 \quad 2 \quad 1$$

$$(a + b)^3 = 1 \quad 3 \quad 3 \quad 1$$

$$(a + b)^4 = 1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$(a + b)^5 = 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

1.6.3 Division of polynomials

In this section we restrict ourselves to polynomials which involve only a single variable, and we shall assume them to be arranged in descending powers of that variable. Let $P(x)$ and $D(x)$ be polynomials of degrees n and r, respectively, where $n \geq r$. Then there exist polynomials $Q(x)$ called the quotient, and $R(x)$ called the remainder such that:

1) $P(x) = D(x)\,Q(x) + R(x)$ for all x

2) The degree of $R(x)$ is less than the degree of $D(x)$

In the examples below we illustrate the method for the division of polynomials.

Example 1 : Divide $(2x^3 + 6x^2 + 10x + 6) \div (2x + 2)$

$$
\begin{array}{rl}
2x^3 + 6x^2 + 10x + 6 & \big| \underline{2x + 2} \\
-\,2x^3 - 2x^2 & \quad x^2 + 2x + 3 \\
\hline
0 \quad + 4x^2 + 10x + 6 & \\
\quad - 4x^2 - 4x & \\
\hline
\qquad\qquad 6x + 6 & \\
\qquad - 6x - 6 & \\
\hline
\qquad\qquad 0 &
\end{array}
$$

$$(2x^3 + 6x^2 + 10x + 6) = (2x + 2)(x^2 + 2x + 3)$$

Example 2 : Divide $(6x^3 - 7x^2 + 14x - 8) \div (3x - 2)$

$$
\begin{array}{rl}
6x^3 - 7x^2 + 14x - 8 & \big| \underline{3x - 2} \\
-\,6x^3 + 4x^2 & \quad 2x^2 - x + 4 \\
\hline
0 \quad - 3x^2 + 14x - 8 & \\
\quad + 3x^2 - 2x & \\
\hline
\qquad\quad 0 \quad + 12x - 8 & \\
\qquad\quad - 12x + 8 & \\
\hline
\qquad\qquad\quad 0 &
\end{array}
$$

$$(6x^3 - 7x^2 + 14x - 8) = (3x - 2)(2x^2 - x + 4)$$

1.7 Simple equations

A simple equation expresses a statement of equality, e.g. $5x = 30$. The process of finding the value of x which thus satisfies the equation is called "solving the equation", in our example:

$$
\begin{aligned}
5x &= 30 \\
x &= 30 \div 5 \\
x &= 6
\end{aligned}
$$

Solve the equation:

$$8x - 10 = 14$$

We can write the equation in this form:

$$8x = 14 + 10$$

and,

$$
\begin{aligned}
8x &= 24 \\
x &= 24 \div 8 \\
x &= 3
\end{aligned}
$$

Properties:

1) If the same number is added to, or subtracted from both sides of an equation, the two sides will again be equal. This is equivalent to transferring a number from one side of an equation to the other, at the same time changing its sign, i.e. change + to - and - to +.

2) If both sides of an equation are multiplied or divided by the same number, the two sides of the new equation will be equal. If the multiplier is -1, both sides change signs.

Solve the equation:

$$4x - \frac{x-2}{3} = 5 + \frac{2x+1}{4}$$

Multiplying throughout by 12, we get:

$$48x - 4(x-2) = 60 + 3(2x+1)$$

Clearing brackets:

$$48x - 4x + 8 = 60 + 6x + 3$$

$$48x - 4x - 6x = 60 + 3 - 8$$
$$38x = 55$$
$$x = \frac{55}{38}$$

1.7.1 Literal equations

It is frequently necessary to solve equations in which the values of the unknown quantities will be found in terms of letters which occur in the equation. Such equations are termed "literal equations".

Solve the equation: $5x - 3a = 2x - 3b$

$$5x - 2x = 3a - 3b$$
$$3x = 3(a-b)$$
$$x = \frac{3(a-b)}{3} = a-b$$

Example:

Solve the equation:

$$\frac{x}{m} - \frac{1}{n} = \frac{3x}{mn}$$

Multiplying throughout by mn we get:

$$\begin{aligned} nx - m &= 3x \\ nx - 3x &= m \\ (n-3)\,x &= m \\ x &= \frac{m}{n-3} \end{aligned}$$

1.8 Simultaneous equations[1/] (two unknowns)

There are cases in which we consider a pair of equations simultaneously in two variables. Such equations are called "simultaneous equations".

∴ Solve the system of equations:

$$\begin{aligned} (1) \quad & 4x + 2y = 42 \\ (2) \quad & 3x + 4y = 44 \end{aligned}$$

For the solution of the above system of equations, we employ the following method:

(a) In equation (1) we obtain the one letter in terms of the other:

$$\begin{aligned} 2y &= 42 - 4x \\ (3) \quad y &= 21 - 2x \end{aligned}$$

(b) By substitution in equation (2) the value of y we get:

$$\begin{aligned} 3x + 4(21-2x) &= 44 \\ 3x + 84 - 8x &= 44 \\ 3x - 8x &= 44 - 84 \\ -5x &= -40 \\ x &= \frac{-40}{-5} = 8 \end{aligned}$$

1/ For the applications of Matrices to simultaneous equations, see Bazigos, G.P. (1975): Applied Fishery Statistics, Vectors and Matrices. FAO/T135,Sup.1

(c) By substituting the value of x in equation (3) we get:

$$y = 21 - 2 \times 8 = 5$$

(d) The solution is $x = 8$ and $y = 5$

These values should be checked by substitution in both of the given equations.

∴ A second method which can be used for solving simultaneous equations is demonstrated below.

Solve the system of equations:

$$\begin{aligned} (1) \quad & 2x + y = 30 \\ (2) \quad & 3x - y = 20 \end{aligned}$$

If the left and right sides of the equations were added respectively we get:

$$\begin{aligned} (2x + 3x) + (y - y) &= 50 \\ 5x + 0 &= 50 \\ 5x &= 50 \\ x &= 10 \end{aligned}$$

By substituting this x value in equation (1) [or (2)] we get:

$$\begin{aligned} 2 \times 10 + y &= 30 \\ y &= 30 - 20 = 10 \end{aligned}$$

The solution is $x = 10$ and $y = 10$

∴ Solve the system of equations:

$$\begin{aligned} (1) \quad & 2x + 3y = 22 \\ (2) \quad & 4x - 2y = 12 \end{aligned}$$

In the above system, elimination of y can be obtained by multiplying both sides of equation (1) by 2, and both sides of equation (2) by 3:

$$\begin{aligned} (3) = (1) \times 2 \quad & 4x + 6y = 44 \\ (4) = (2) \times 3 \quad & 12x - 6y = 36 \quad (+) \\ & 16x + 0 = 80 \\ & x = \frac{80}{16} = 5 \end{aligned}$$

Substituting this value of x in equation (1) we get:

$$\begin{aligned} 2 \times 5 + 3y &= 22 \\ 3y &= 22 - 10 = 12 \\ y &= 4 \end{aligned}$$

The solution to the system is $x = 5$ and $y = 4$

It should be noted that the general expression of a pair of simultaneous linear equations is:

$$\begin{aligned} a_1x + b_1y + c_1 &= 0 \quad , \quad a_1 \neq 0,\ b_1 \neq 0 \\ a_2x + b_2y + c_2 &= 0 \quad , \quad a_2 \neq 0,\ b_2 \neq 0 \end{aligned}$$

Exercises:

1) Find the values of z_1 and z_2 which will satisfy the equations:

$$\begin{aligned} 1.25\, z_1 - 0.75\, z_2 &= 1 \\ 0.25\, z_1 + 1.25\, z_2 &= 17 \end{aligned}$$

2) $$\frac{x}{3} + \frac{y}{2} = \frac{2x}{3} - \frac{y}{6} = 7$$

3) $$\begin{aligned} 7x + 3(y-3) &= 5(x+y) \\ 7(x-1) - 6y &= 5(x-y) \end{aligned}$$

4) $$\begin{aligned} 3P + 10Q &= 8.6 \\ 2P - 5Q &= 2.0 \end{aligned}$$

5) $$\frac{x-1}{3} + \frac{y+2}{2} = 3$$

$$\frac{1-x}{6} - \frac{y-4}{2} = \frac{1}{2}$$

6) $$\begin{aligned} 8.2w - 1.5u &= 7.020 \\ 2.5w + 3.7u &= 13.365 \end{aligned}$$

7) $$3a - \frac{b}{3} = \frac{31}{2}$$

$$\frac{a}{2} + b = 1$$

1.9 Graphical presentation of magnitudes

The graphical method developed on scientific and mathematical lines is also largely employed in mathematics and science, in order to portray the behaviour of a magnitude or certain important principles of the surveyed magnitudes[1/].

2.0 The straight line

Consider the linear equation $y = x$ (in this equation, x and y are of the first degrees, i.e. contains no higher powers than the first of x and y). This equation represents a straight line passing through the origin.

Similarly, the equation $y = 2x$ represents a straight line passing through the origin.

The following tables (Tables 1,2) give the values of co-ordinates for the two equations:

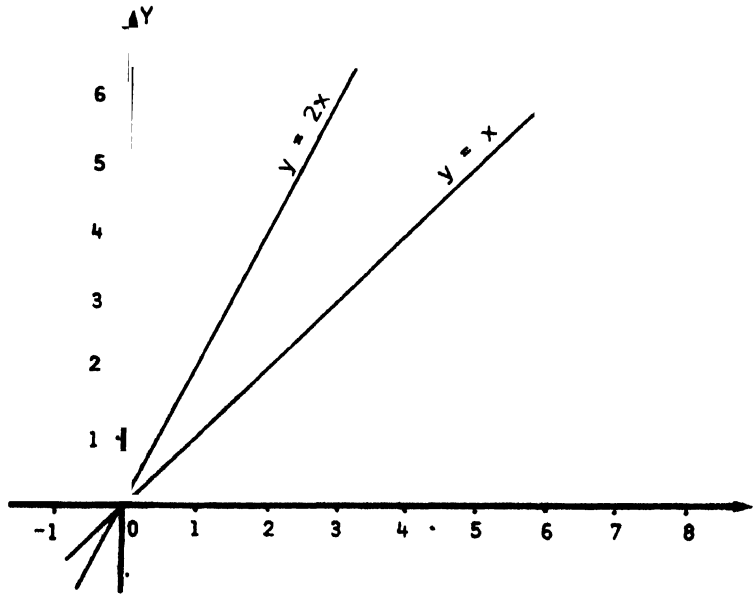
Table 1 (y = x)

x	0	1	2	3	4
y	0	1	2	3	4

Table 2 (y = 2x)

x	0	1	2	3	4
y	0	2	4	6	8

By using the tabulated values (Tables 1,2) we draw the graphs of the respective lines:



1/ See: Bazigos, G.P. (1974) - Applied Fishery Statistics, FAO-FIPS/T135,

Substituting this value of x in equation (1) we get:

$$\begin{aligned} 2 \times 5 + 3y &= 22 \\ 3y &= 22 - 10 = 12 \\ y &= 4 \end{aligned}$$

The solution to the system is $x = 5$ and $y = 4$

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3) $$\begin{aligned} 7x + 3(y - 3) &= 5(x + y) \\ 7(x - 1) - 6y &= 5(x - y) \end{aligned}$$

4) $$\begin{aligned} 3P + 10Q &= 8.6 \\ 2P - 5Q &= 2.0 \end{aligned}$$

5) $$\frac{x - 1}{3} + \frac{y + 2}{2} = 3$$

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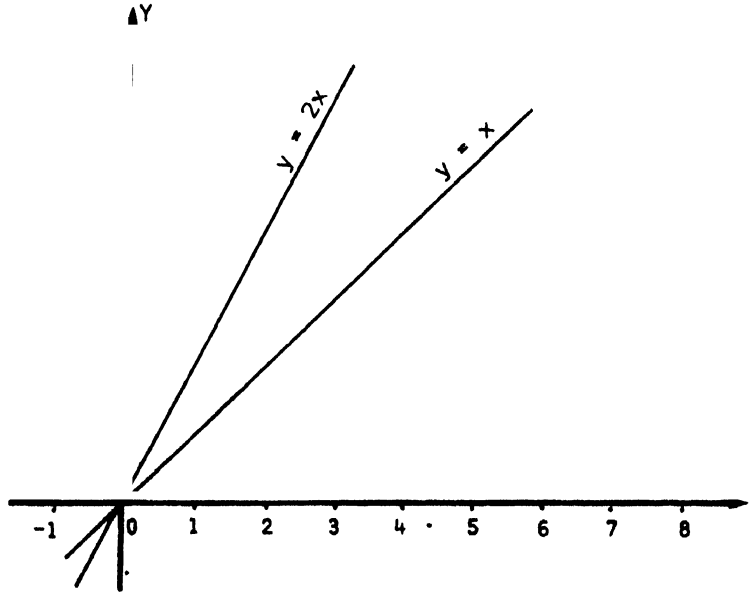
y	0	1	2	3	4

Table 2 (y = 2x)

x	0	1	2	3	4
y	0	2	4	6	8

By using the tabulated values (Tables 1,2) we draw the graphs of the respective lines:



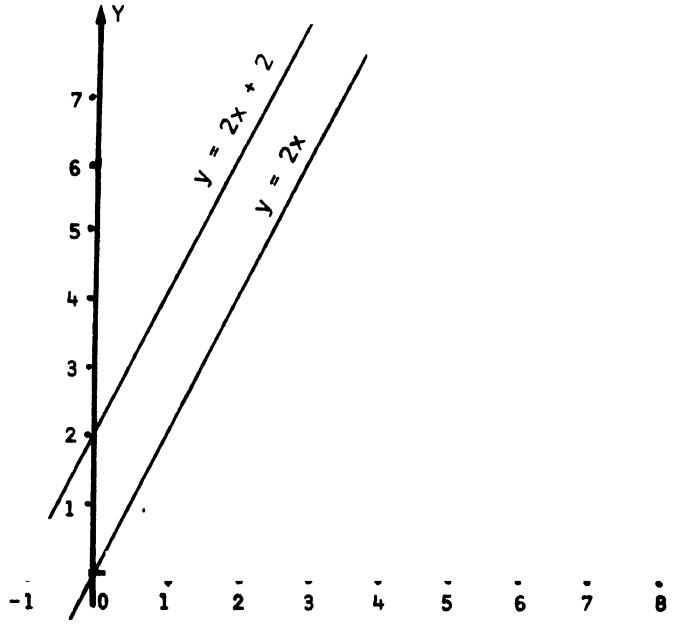
1/ See: Bazigos, G.P. (1974) - Applied Fishery Statistics, FAO-FIPS/T135.

Now consider the equation $y = 2x+2$. This represents a straight line parallel to the line $y = 2x$ and cutting the axis of Y at distance 2 from the origin (intercept on the Y-axis):

$(y = 2x + 2)$

x	0	1	2	3	4
y	2	4	6	8	10

Using the table of the above values, the straight line is shown in the graph below:



The equation of a straight line can be expressed in general terms by the equation 1/:

$$y = ax + b$$

It should be noted that a straight line is fixed by two points, but in drawing it from its equation, three points should always be taken to check accuracy.

Example: Draw the graph of the equation:

$$2y - 6x = 4$$
$$2y = 6x + 4$$
$$y = 3x + 2$$

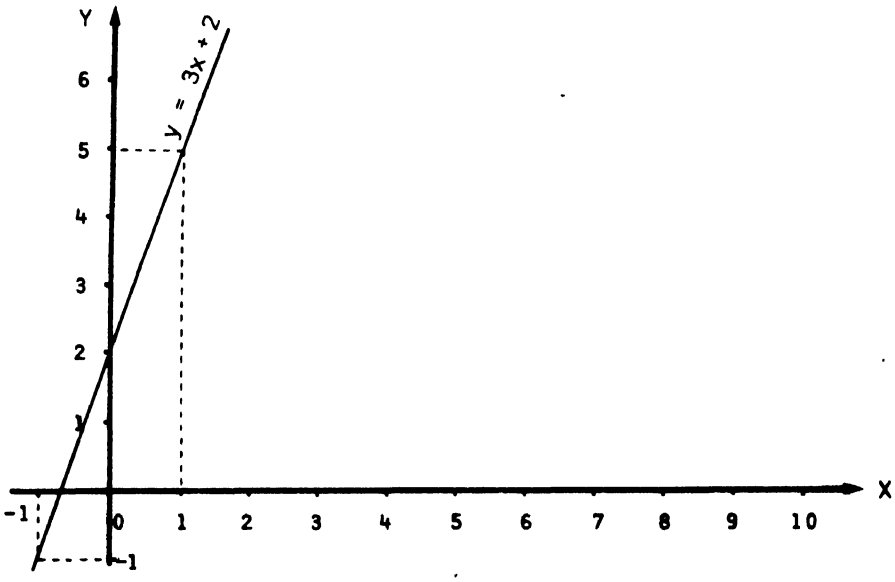
Giving suitable values to x, the corresponding values of y can be calculated:

$(y = 3x + 2)$

x	-2	-1	0	1	2	3
y	-4	-1	2	5	8	11

1/ Other expressions used are: $y = a + bx$, $y = mx + b$, etc.

Using the table of the above values, the straight line is shown in the graph below:



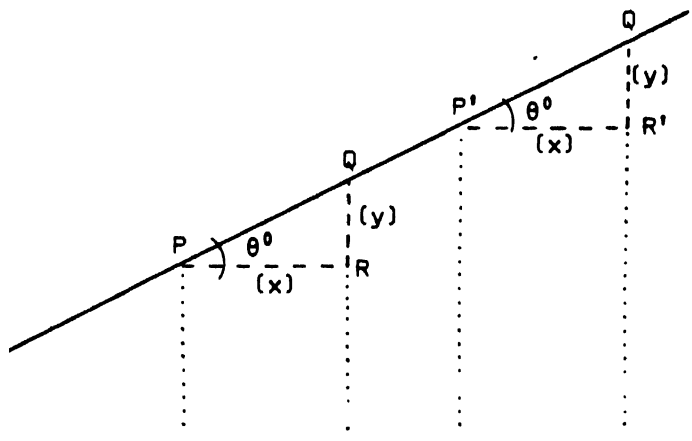
Gradient of a straight line

Consider two points P,Q on a straight line (Figure below). Let PR and QR be parallel to OX and OY respectively. Then the ratio $\frac{QR}{PR}$ is constant for all pairs of points on the line.

This ratio $\frac{QR}{PR}$ constant for any two points on the straight line, is defined as the "gradient" of the straight line:

Gradient of straight line 1/ (a) = $\frac{QR}{PR}$ = $\frac{\text{increase in Y}}{\text{increase in X}}$

: rate of increase of Y with respect to X



1/ Also, $\frac{QR}{PR}$ = tan θ, where θ is the angle between the line and the positive direction of the X-axis.

From the above discussion one can see that a straight line is a graph which has a constant gradient.

In summary, we have:

(a) Lines passing through the origin: $y = ax$.
If a is positive, the line will pass through the first and third quadrants. If a is negative, the line will pass from the second and fourth quadrants.

(b) Lines not passing through the origin: $y = ax + b$.
In the equation, a is the gradient of the line, and b is the intercept on the Y-axis.

2.2 Equations of straight lines

Exercise 1:

Find the equation of a straight line passing through the point (2,-1) and having gradient 3.

Solution :- The general form of an equation having gradient 3 is:

$$y = 3x + b$$

As the line passes through the point with co-ordinates $x = 2$, $y = -1$, we substitute these values in the above equation:

$$-1 = 3 \times 2 + b$$

$$b = -7$$

The equation of the line is:

$$y = 3x - 7 \quad \text{or} \quad y - 3x + 7 = 0$$

Exercise 2:

Find the equation of a line passing through the points P(-1,2) and Q(2,4).

Solution :-

(i) The co-ordinates of the points are
$P(x_1 = -1,\ y_1 = 2)$, $Q(x_2 = 2,\ y_2 = 4)$

(ii) The gradient of the line will be:

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{2-(-1)} = \frac{2}{3}$$

(iii) The general form of the equation of the line passing through two points is:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Substituting the respective values in the above equation we get:

$$\frac{y-2}{4-2} = \frac{x-(-1)}{2-(-1)}$$

$$\frac{y-2}{2}$$

$$3y - 6 = 2x + 2$$

$$3y = 2x + 2 + 6$$

$$y = \tfrac{2}{3}x + \tfrac{8}{3}, \quad (a = \tfrac{2}{3}, \quad b = \tfrac{8}{3})$$

Parallel lines

Exercise:

Find the straight line parallel to the line $3y - x = 2$ and passing through the point $P(3,-2)$.

Solution :-

(i) The equation of the given line can be written in the form,

$$3y = x + 2$$

or,

$$y = \tfrac{1}{3}x + \tfrac{2}{3}, \quad (a = \tfrac{1}{3}, \quad b = \tfrac{2}{3})$$

(ii) The line required should have the same gradient as the given line ($a = \frac{1}{3}$) and should pass through the point P with co-ordinates $x_1 = 3$, $y_1 = -2$. The general form of the equation of the line is:

$$(y - y_1) = a(x - x_1)$$

Substituting, we get:

$$(y - (-2)) = \tfrac{1}{3}(x - 3)$$

$$y + 2 = \tfrac{1}{3}(x - 3)$$

$$3y + 6 = x - 3$$

$$y = \tfrac{1}{3}x - 3 \quad \text{or} \quad 3y - x + 9 = 0$$

Perpendicular lines

If $y = ax + b$ and $y' = a'x + b'$ represent equations of two perpendicular lines, then the following equality is valid:

$$a \times a' = -1$$

$$a \cdot a' = -1$$

Exercise:

Find the equation of a straight line perpendicular to the line $y - 3x = 2$ and passing through the origin [P(0,0)].

Solution :-

(i) The equation of the given line can be written:

$$y = 3x + 2, \quad a = 3$$

(ii) Gradient of the perpendicular line:

$$a \times a' = -1$$

Substituting, we get:

$$3 \times a' = -1, \qquad a' = -\tfrac{1}{3}$$

(iii) The equation of the required line passing through the origin will be:

$$(y - 0) = -\tfrac{1}{3}(x - 0)$$

$$y = -\tfrac{1}{3}x \quad \text{or} \quad 3y + x = 0$$

2.3 Graphical solution of simultaneous equations (two unknowns)

The conclusions reached in the previous section (2.0) can be used to solve simultaneous equations of the first degree.

Example:

Solve the system of equations:

(1) $3x - 2y = 7$

(2) $x + 2y = 5$

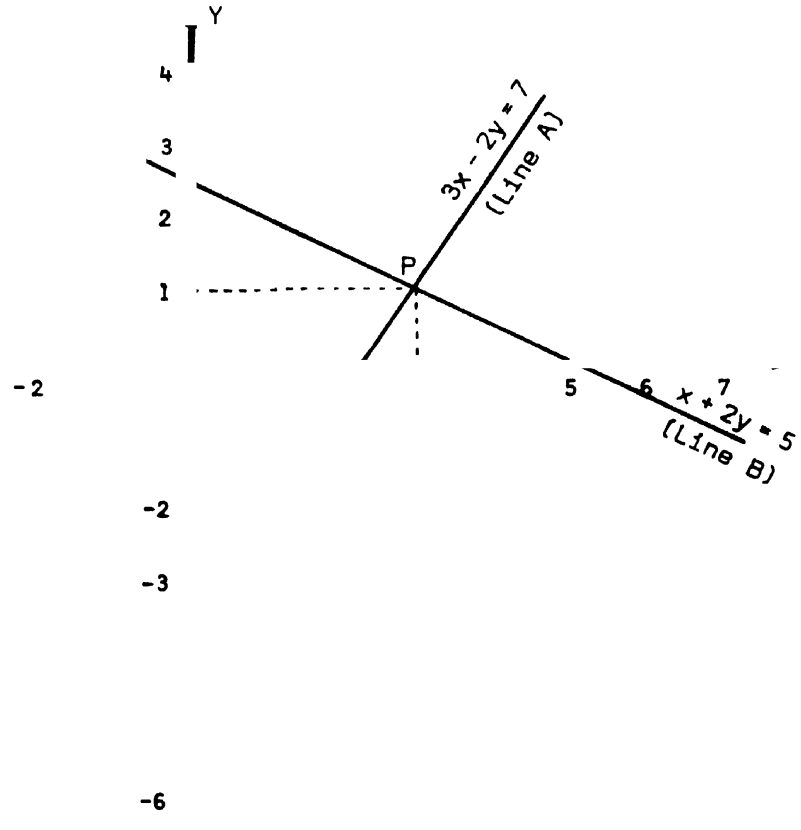
(a) Draw the graph of $3x - 2y = 7$ or $y = 1.5x - 3.5$
The table of co-ordinates is as follows:

x	0	1	2	3	4
y	-3.5	-2	-.5	1	2.5

(b) Draw the graph of $x + 2y = 5$ or $y = -0.5x + 2.5$
The table of co-ordinates is as follows:

x	0	1	2	3	4
y	2.5	2	1.5	1	.5

(c) By using the above tabulated values we draw the graphs of the respective lines.



(d) Line A contains all those points of which the co-ordinates satisfy the equation $3x - 2y = 7$.
Line B contains all those points of which the co-ordinates satisfy the equation $x + 2y = 5$.
There is one point, and one point only, where the co-ordinates satisfy both equations. That point is P, the interception of the two lines. The co-ordinates of P, by inspection, are (3,1).
The solution of the system of the equations is:

$$x = 3, \quad y = 1$$

2.4 Simultaneous linear equations in three unknowns

The general expression for such a system of equations is:

1) $a_1x + b_1y + c_1z + d_1 = 0$ Not all $a_1, b_1, c_1 = 0$

2) $a_2x + b_2y + c_2z + d_2 = 0$ Not all $a_2, b_2, c_2 = 0$

3) $a_3x + b_3y + c_3z + d_3 = 0$ Not all $a_3, b_3, c_3 = 0$

A solution[1/] of the above system is an ordered triple (x, y, z) which satisfies all three equations.

Solve:

$$\begin{aligned} 1)\quad & 2x - y + 3z = -9 \\ 2)\quad & x + 3y - z = 10 \\ 3)\quad & 3x + y - z = 8 \end{aligned}$$

(a) We eliminate x between the (1) and (2) equations, and between the (1) and (3) equations:

(i)

$$\begin{aligned} 1)\quad & 2x - y + 3z = -9 \\ 2)\quad & x + 3y - z = 10 \quad (-2) \end{aligned}$$

$$\begin{aligned} 1)\quad & 2x - y + 3z = -9 \\ 2)\quad & -2x - 6y + 2z = -20 \quad (+) \\ 4)\quad & -7y + 5z = -29 \end{aligned}$$

(ii)

$$\left.\begin{aligned} 1)\quad & 2x - y + 3z = -9 \\ 2)\quad & 3x + y - z = 8 \end{aligned}\right\} \begin{aligned} &(-3) \\ &(2) \end{aligned}$$

$$\begin{aligned} 1)\quad & -6x + 3y - 9z = 27 \\ 3)\quad & 6x + 2y - 2z = 16 \\ 5)\quad & 5y - 11z = 43 \end{aligned}$$

(iii) The result, equations (4) and (5), together with the unchanged first equation, is the equivalent system:

$$\begin{aligned} 1)\quad & 2x - y + 3z = -9 \\ 4)\quad & -7y + 5z = -29 \\ 5)\quad & 5y - 11z = 43 \end{aligned}$$

(b) We eliminate y between the equations (4) and (5):

$$\left.\begin{aligned} 4)\quad & -7y + 5z = -29 \\ 5)\quad & 5y - 11z = 43 \end{aligned}\right\} \begin{aligned} &(5) \\ &(7) \end{aligned}$$

$$\begin{aligned} 4)\quad & -35y + 25z = -145 \\ 5)\quad & 35y - 77z = 301 \\ & -52z = 156 \end{aligned}$$

1/ For the applications of matrices to simultaneous equations, see:
Bazigos, G.P. (1975) - Applied Fishery Statistics, FAO-FIPS/T135,Sup.1

(c) Putting $z = -3$ in equation (4) enables us to find the value of y:

$$4) \quad -7y + 5x(-3) = -29$$
$$-7y = -14$$

(d) Putting $z = -3$ and $y = 2$ in equation (1) enables us to find the value of x:

$$1) \quad 2x - 2 + 3(-3) = -9$$
$$2x = 2$$
$$1$$

Hence, the solution is (1, 2, -3).

2.5 Factoring

In factoring we seek to undo the process of multiplication. We are given a polynomial and are asked to discover how it can be expressed as a product of other polynomials, called its factors.

The problem of factoring a general polynomial can be quite a complicated affair. We have a more limited objective here, namely, to factor certain simple polynomials by methods which are straight forward:

(a) Removal of a common factor(s) (examples):

Example 1 : Find the factors of $ax + bx$

$$ax + bx = x(a+b)$$

Example 2 : Find the factors of $6a^3 + 3a^2c$

$$6a^3 + 3a^2c = 3a^2(2a+c)$$

Example 3 : Find the factors of $x^3 - 3x^2 - x$

$$x^3 - 3x^2 - x = x(x^2 - 3x - 1)$$

e 4 : Find the factors of $\frac{bc^2}{6} - \frac{abc}{9}$

$$bc^2 \quad \frac{abc}{9} = \frac{bc}{3}\left(\frac{c}{2} - \frac{a}{3}\right)$$

(b) Binomial factors. We know that:

$$(x+a)(y+b) = x(y+b) + a(y+b)$$
$$= xy + bx + ay + ab$$

If we require the factors of $xy + bx + ay + ab$ and similar expressions, we must work backward, through the steps shown above, i.e.

(i) We group them in the form
$(xy + bx) + (ay + ab)$

(ii) Taking out the common factor
$x(y+b) + a(y+b)$

(iii) Then $(y+b)$ is a common factor of the above terms
$(x+a)(y+b)$

Example 1: Find the factors of $x^2 + px + qx + pq$

$$x^2 + px + qx + pq = (x^2 + px) + (qx + pq)$$
$$= x(x+p) + q(x+p)$$
$$= (x+q)(x+p)$$

Example 2: Find the factors of $ax^2 + a^2x - ab - bx$

$$ax^2 + a^2x - ab - bx = (ax^2 + a^2x) - (ab + bx)$$
$$= ax(x+a) - b(a+x)$$
$$= (ax - b)(x+a)$$

(c) Trinomials with integral coefficients.

1) Trinomials of the form $x^2 + bx + c$:

We know that, $(x+a)(x+b) = x(x+b) + a(x+b)$
$$= x^2 + bx + ax + ab$$
$$= x^2 + (b+a)x + ab$$

This result suggests a quick way of obtaining the factors of a trinomial of the above form. The coefficient of x is the sum of a and b. The last term is their product. Thus, we have:

$$(x+6)(x+5) = x^2 + (5+6)x + 5 \times 6$$
$$x^2 + 11x + 30$$

Reversing the process we must now consider how to find the factors of $x^2 + 11x + 30$. In finding the factors of an expression such as $x^2 + 11x + 30$, we must find by trial two numbers, the product of which is 30 and the sum 11. They are $+5$ and 6:

$$x^2 + 11x + 30 = (x+6)(x+5)$$

Example 1: Factorize $x^2 + 3x + 2$

$$x^2 + 3x + 2 = (x+1)(x+2)$$

Example 2: Factorize $y^2 - 13y + 30$

$$y^2 - 13y + 30 = (y-10)(y-3)$$

2) Trinomials of the form $ax^2 + bx + c$:

When the coefficient of the first term of the trinomial is not unity we have:

$$(px+a)(qx+b) = pqx^2 + (pb+aq)x + ab$$

Example: $(2x+5)(3x+4)$

$$(2x+5)(3x+4) = (2\times 3)x^2 + (2\times 4 + 5\times 3)x + 5\times 4$$
$$6x^2 + 23x + 20$$

(p) (a) (q) (b)

From the above analysis, it is indicated that the coefficient of the middle term of the trinomial can be obtained by multiplying as shown in the arrows above.

Reversing the process, we can now find the factors of $6x^2 + 23x + 20$:

$$6x^2 + 23x + 20 = (px+a)(qx+b)$$

These factors are obtained by trial.

We write down pairs of factors systematically and find the middle term as shown above:

(1) $(2x+5)(3x+4)$, $(8 + 15 = 23)$

(2) $(2x+4)(3x+5)$, $(10+12 = 22)$

(3) $(2x+10)(3x+2)$, $(4 + 30 = 34)$

(4) $(2x+2)(3x+10)$, $(20 + 6 = 26)$

In (1) the coefficient of x is 23 (correct). Thus,

$$6x^2 + 23x + 20 = (2x+5)(3x+4)$$

Example 1: Find the factors of $2x^2 + 7x + 3$

The positive sign of the last term indicates that the factors will have the same signs in the second terms.

In this example, the possibilities are:

(1) $(2x+3)\ (x+1)$

(2) $(2x+1)\ (x+3)$

In (1) the coefficient of x is 5.
In (2) the coefficient of x is 7 (correct). Thus,

$$2x^2 + 7x + 3 = (2x+1)\ (x+3)$$

Example 2: Find the factors of $6x^2 + 17x - 3$

The minus sign of the last term indicates that the factors will have opposite signs in the second terms. This increases the number of possible pairs of factors.

(1) $(2x-1)\ (3x+3)$, $(6 - 3 = 3)$

(2) $(2x+1)\ (3x-3)$, $(-6+3 = -3)$

(3) $(6x-1)\ (x+3)$, $(18-1 = 17)$

(4) $(6x+1)\ (x-3)$, $(-18+1 = -17)$

The pairs of (3) is the one required

$$6x^2 + 17x - 3 = (6x-1)\ (x+3)$$

2.6 Algebraic fractions

An algebraic fraction is the quotient of two algebraic expressions. Examples of these are:

(1) $\dfrac{3x^2+4a}{x+1}$, (2) $\dfrac{\sqrt{x^3-1}+4x}{\sqrt[3]{x+7}}$, (3) $\dfrac{\sqrt{x^2}+\sqrt{3}x}{2/(x+5)}$

2.6.1 Simplification of fractions

Algebraic fractions are subject to the same laws as arithmetical fractions. Simplification of algebraic fractions is desirable before putting them into some further calculations. The best way of doing this is nothing more than the familiar "reducing to lowest terms". This is obtained by dividing both numerator and denominator by the same number:

$$\frac{a}{b} = \frac{a \times c}{b \times c} \qquad \text{and} \qquad \frac{a}{b} = \frac{a \div c}{b \div c}$$

Example 1:

Simplify $\frac{a - b}{a^2 - b^2}$:

$$\frac{a - b}{a^2 - b^2} = \frac{(a - b)}{(a + b)(a - b)}$$

Example 2:

Simplify $\frac{3a^3 - 12a^2b + 12ab^2}{6a^4 + 18a^3b - 60a^2b^2}$

$$\frac{3a^3 - 12a^2b + 12ab^2}{6a^4 + 18a^3b - 60a^2b^2} = \frac{3a\,(a^2 - 4ab + 4b^2)}{6a^2(a^2 + 3ab - 10b^2)}$$

$$= \frac{3a\,(a - 2b)^2}{6a^2(a + 5b)(a - 2b)}$$

$$\frac{3a\,(a - 2b)(a - 2b)}{6a^2(a + 5b)(a - 2b)} = \frac{a - 2b}{2a(a + 5b)}$$

Example 3:

Simplify $\frac{x^2 - 5x + 6}{x^2 - 4x + 3}$.

$$\frac{x^2 - 5x + 6}{x^2 - 4x + 3} = \frac{(x - 3)(x - 2)}{(x - 3)(x - 1)}$$

Example 4:

Simplify $\frac{x^2 + 4x + 4}{x^2 + 4x + 3}$

$$\frac{x^2 + 4x + 4}{x^2 + 4x + 3} = \frac{(x + 2)(x + 2)}{(x + 1)(x + 3)}$$

This does not simplify the algebraic fraction since the numerator and denominator have no common factor.

∴ It is understood that we cannot assign values to any variable, in algebraic fractions, which make the denominator equal to zero. Thus, in the example:

$$\frac{3x^2 + 4a}{x + 1}$$

We exclude $x = -1$.

It should also be noted that two algebraic fractions involving the variable x are equal, if and only if they have the same numerical values when x is put equal to any number for which both fractions are defined.

2.6.2 Multiplication and division

The multiplication of arithmetic fractions follows the rule:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

We also know that division follows the rule:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

These rules apply equally well to algebraic fractions. When a,b,c and d are polynomials, it is desirable to factor them if possible so that simplifications in the final answer can be made easily.

Example 1:

$$\frac{x^2 - y^2}{4y} \times \frac{2y}{x + y}$$

$$= \frac{(x + y)(x - y)}{4y} \times \frac{2y}{(x + y)} = \frac{x - y}{2}$$

Example 2:

$$\frac{x}{x + 1} \div \frac{x^2}{x^2 - 1}$$

$$= \frac{x}{x + 1} \times \frac{x^2 - 1}{x^2} = \frac{x}{x + 1} \times \frac{(x + 1)(x - 1)}{x^2}$$

$$= \frac{x - 1}{x}$$

Example 3:

$$\frac{x^2 + 3x + 2}{x^2 - x - 2} \times \frac{x^2 + x - 2}{x^2 + 4x + 4} \div \frac{x^2 - \cdot}{x^2 + x - 6}$$

$$\frac{(x + 2)(x + 1)}{(x - 2)(x + 1)} \times \frac{(x + 2)(x - 1)}{(x + 2)(x + 2)} \times \frac{(x - 2)(x + 3)}{x(x - 1)}$$

$$x + 3$$

2.6.3 Addition and subtraction

The addition and subtraction is a straightforward application of the formula below:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

Example 1:

$$\frac{2x - 1}{x + 3} + \frac{x^2}{3x - 1}$$

$$\frac{(2x - 1)(3x - 1) + (x + 3)x^2}{(x + 3)(3x - 1)}$$

$$x^3 + 9x^2 - 5x + 1$$
$$3x^2 + 8x - 3$$

Example 2:

$$\frac{a}{a - b} - \frac{a^2}{a^2 - b^2}$$

$$- \frac{}{a - b} - \frac{}{(a + b)(a - b)}$$

The least common denominator is: $(a + b)(a - b)$.

∴ Fraction $- \dfrac{a(a + b) - a^2}{(a + b)(a - b)}$

$$\frac{a^2 + ab - a^2}{(a + b)(a - b)}$$

$$\frac{ab}{(a + b)(a - b)}$$

Example 3: $\dfrac{x - 1}{x^2 - x - 2} - \dfrac{x + 2}{x^2 + 4x + 3}$

Factorising the fractions, they become:

$$= \frac{x - 1}{(x - 2)(x + 1)} - \frac{x + 2}{(x + 3)(x + 1)}$$

The least common denominator is: $(x + 1)(x - 2)(x + 3)$.

$$\therefore \text{ Fraction } = \frac{(x - 1)(x + 3) - (x + 2)(x - 2)}{(x + 1)(x - 2)(x + 3)}$$

$$= \frac{(x^2 + 2x - 3) - (x^2 - 4)}{(x + 1)(x - 2)(x + 3)}$$

$$= \frac{2x + 1}{(x + 1)(x - 2)(x + 3)}$$

Example 4:

Convert the formula $\dfrac{1}{A} = \dfrac{1}{A_1} - \dfrac{1}{A_2}$ into one in which the subject is A:

$$\frac{1}{A} = \frac{A_2 - A_1}{A_1 A_2} \qquad \text{and} \qquad A = \frac{A_1 A_2}{A_2 - A_1}$$

2.6.4 Compound fractions

The operations of adding, subtracting, multiplying and dividing fractions can be combined in various ways:

Example:

$$\frac{\frac{1}{x+1} - \frac{1}{x-1}}{\frac{1}{x+1} + \frac{1}{x-1}} = \frac{\frac{(x-1)-(x+1)}{(x+1)\ (x-1)}}{\frac{(x-1)+(x+1)}{(x+1)\ (x-1)}}$$

$$\frac{\frac{-2}{(x+1)\ (x-1)}}{\frac{2x}{(x+1)\ (x-1)}}$$

$$\frac{-2}{(x+1)\ (x-1)} \div \frac{2x}{(x+1)\ (x-1)}$$

$$\frac{-2}{(x+1)\ (x-1)} \times \frac{(x+1)\ (x-1)}{2x}$$

2.7 Simple equations involving algebraical fractions

The following examples will illustrate the methods employed:

Example 1:

Solve the equation $\frac{3}{x-2} = \frac{5}{x-1}$

The least common denominator is $(x-1)(x-2)$

Multiply both sides by this, we get:

$$\frac{3(x-1)(x-2)}{(x-2)} = \frac{5(x-1)(x-2)}{(x-1)}$$

$$3(x-1) = 5(x-2)$$

$$3x - 3 = 5x - 10$$
$$3x - 5x = -10 + 3$$
$$-2x = -7$$
$$x = 3.5$$

Example 2:

Solve the equation $\frac{1}{m-3} + \frac{1}{m-2} = \frac{2}{m}$

The least common denominator is $m(m-2)(m-3)$

Multiply both sides by this, we get:

$$\frac{m(m-3)(m-2)}{m-3} + \frac{m(m-3)(m-2)}{m-2} = \frac{2m(m-2)(m-3)}{m}$$

$$m(m-2) + m(m-3) = 2(m-2)(m-3)$$

$$m^2 - 2m + m^2 - 3m = 2(m^2 - 5m + 6)$$
$$2m^2 - 5m = 2m^2 - 10m + 12$$

$$-5m + 10m = 12$$
$$5m = 12$$
$$m = 2.4$$

2.8 Graphs of quadratic functions

2.8.1 Introduction

Letters used in algebraic expressions represent two different kinds of numbers:

1) Constant numbers, i.e. numbers which remain unchanged in the varying cases to which the formula applies

2) Other numbers which represent variable quantities

Further variables are seen to be of two kinds:

a) A dependent variable is a variable which depends on another variable for its value

b) An independent variable is the variable upon which the first one depends

In the expression: $y = mx + b$ m,b are constants and y,x are variables. Further, y is the dependent variable and x is the independent variable.

If a quantity y is related to a quantity x so that for every value which might be assigned to x there is a corresponding value of y, then y is a function of x.

In plotting graphs which show how one quantity varies as another varies, the independent variable is always measured on the X-axis and the dependent one on the Y-axis.

Every function has a distinctive graph, which will be a curve or a straight line, and by which it can be identified.

If the function is of the first degree and does not involve any algebraic fraction, then the graph will be a straight line. If, however, the function involves a higher power of x, such as x^2, x^3 etc., or involves an algebraic fraction such as $y = \frac{1}{x}$, the graph will be a regular curve, the shape of which will differ with the nature of the function.

2.8.2 Graphs of a second degree function

The graph of a function is the set of points in the plane whose ordered pairs of co-ordinates are elements of the solution set of the function.

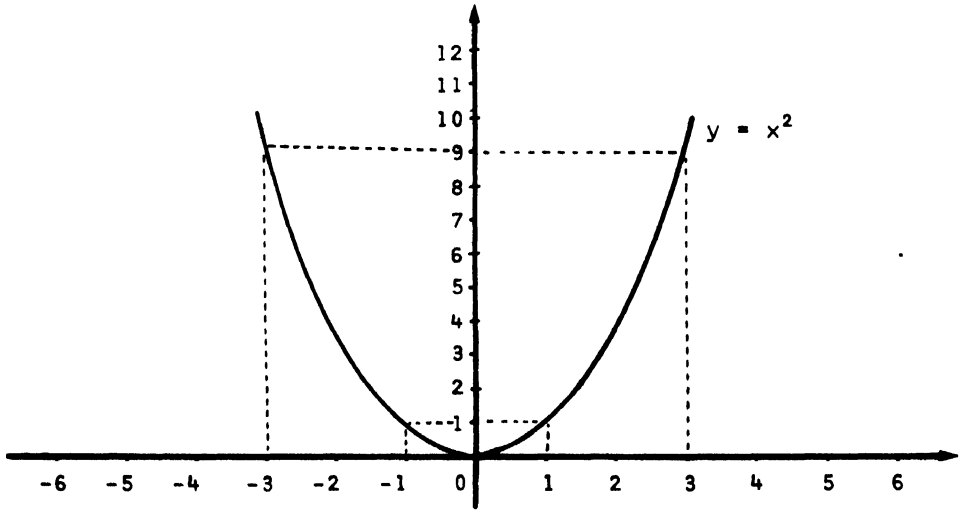
The usual procedure for plotting the graph of an equation of second degree is to compute a small number of points on it and then to connect these with a smooth curve.

The simplest form of a second degree function is that which is expressed by $y = x^2$.

To plot the graph of the above function we assign values to x, calculate the corresponding values of x^2 or y and tabulate them as follows:

(1)	x	-3	-2.5	-2	-1.5	-1	0	1	1.5	2	2.5	3
(2)	y	9	6.25	4	2.25	1	0	1	2.25	4	6.25	9
(3)	$-x^2$	-9	-6.25	-4	-2.25	-1	0	-1	-2.25	-4	-6.25	-9

The points (1,2 above) when plotted appear to lie on a smooth curve, as shown in the figure below. The curve is called a parabola.



Some properties of the curve $y = x^2$ are as follows:

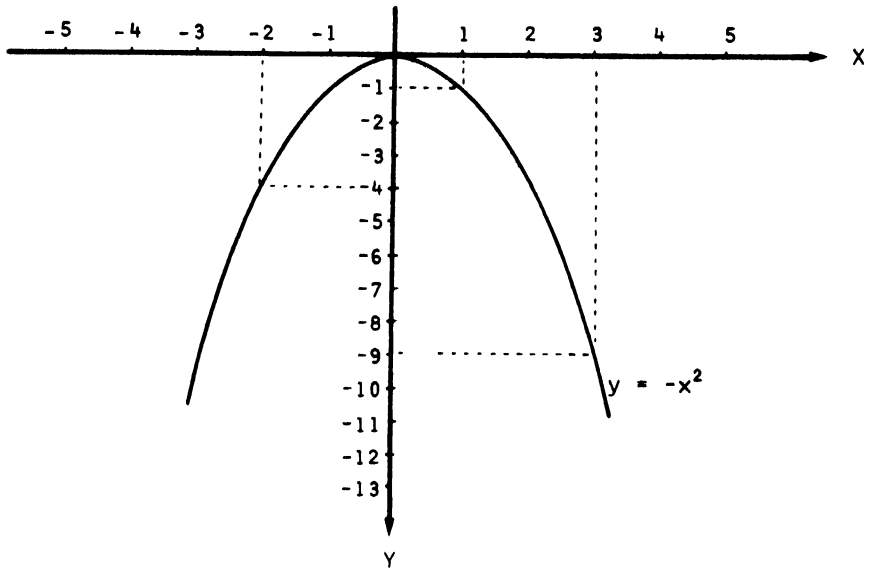
1) The symmetry of the curve: positive and negative values of x produce equal values for y. The curve is therefore said to be symmetrical about OY, and OY is called the axis of symmetry.

2) The minimum value of the curve is 0 at the origin. The curve is said to have a turning point at the origin.

3) The slope of the curve is not constant, as with a straight line, but increases from point to point as x increases. The gradient is clearly a function of x, since its value depends on the value of x.

4) The curve may be used to read off the square of any number within the range of plotted values, and also to conversely determine square roots.

The curve of $y = -x^2$: The points defined by the horizontal columns (1) and (3) above when plotted appear to lie on a smooth curve, as shown in the figure below.



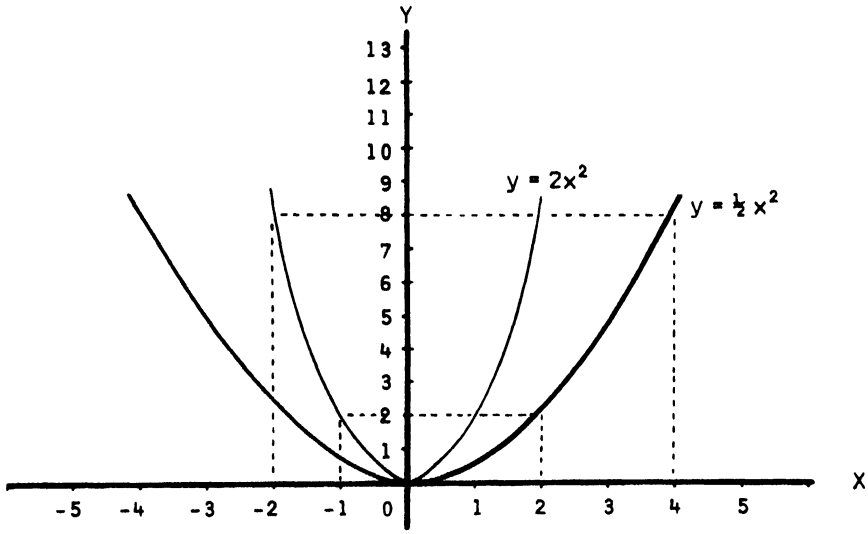
The shape of the curve $y = -x^2$ is the same as that of the curve $y = x^2$, but inverted. The curve has a maximum point at the origin.

The curve of $y = ax^2$: The curves are parabolas differing from $y = x^2$ only by having different slopes (a is any number).

The horizontal columns below give the pair of numbers for some points of the curves $y = 2x^2$ and $y = \frac{1}{2}x^2$.

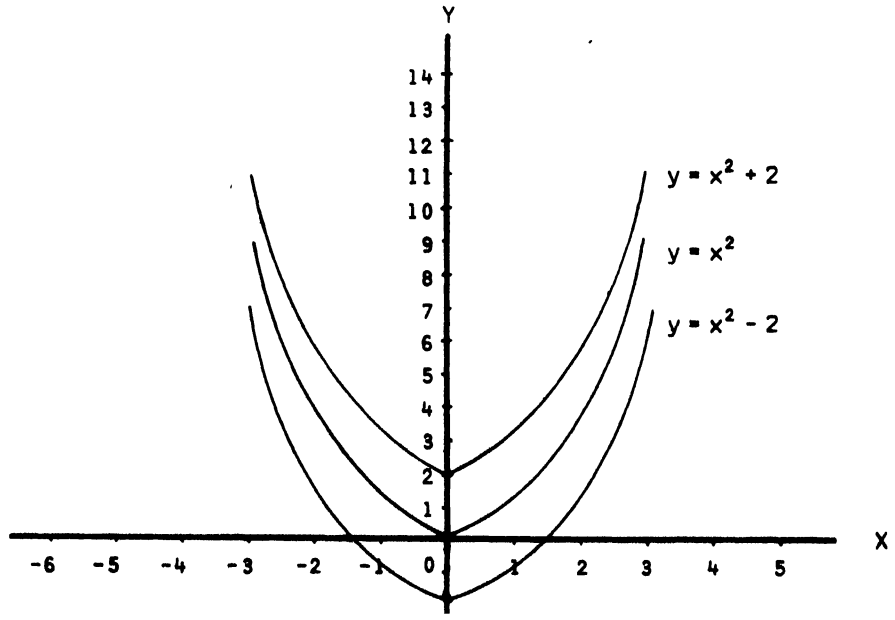
(1)	x	-3	-2	-1	0	1	2	3
(2)	$2x^2$	18	8	2	0	2	8	18
(3)	$\frac{1}{2}x^2$	4.5	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	4.5

Points (1) and (2) above, when plotted, define the curve $y = 2x^2$; and
Points (1) and (3) above, when plotted, define the curve $y = \frac{1}{2}x^2$.



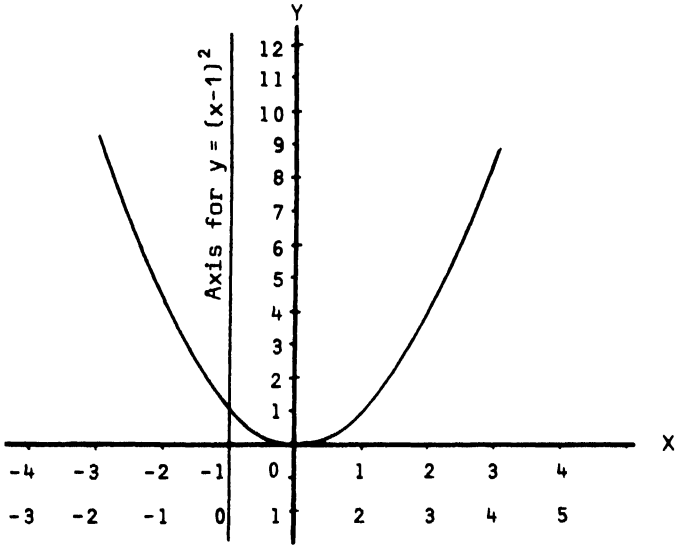
When a is negative, we get the corresponding set of curves similar to $y = -x^2$.

The curves of $y = x^2 \pm c$ where c is any number are the same as the corresponding set of curves $y = x^2$ raised or lowered by c-units on the Y-axis, according to the sign of c.

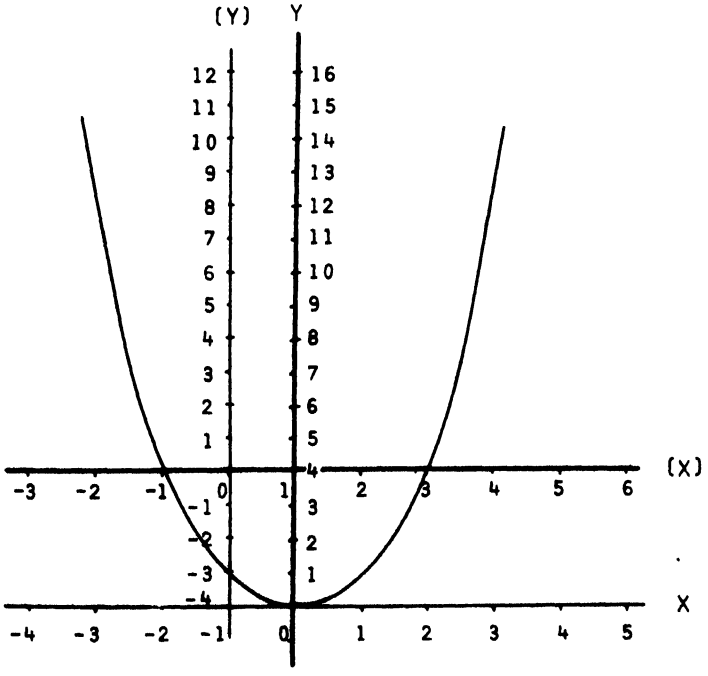


On the examination of the above graph, it will be seen that for the curve $y = x^2 + 2$ the X-axis is two units below the lowest point of the Curve. Consequently, the curve of $y = x^2$ can be changed into that of $y = x^2 + 2$ by drawing a new X-axis two units below the original and renumbering the Y-axis. Similarly, $y = x^2$ becomes $y = x^2 - 2$ by drawing a new X-axis two units above the original, and again renumbering the units of the Y-axis.

The curve of $y = (x-1)^2$: The curve of $y = x^2$ will be changed into the curve of $y = (x-1)^2$ by moving the Y-axis one unit to the left and renumbering the X-axis.



Similarly, if the Y-axis is moved one unit to the right, we should have the curve $y = (x+1)^2$, and so on, for similar functions.



The graph of $y = (x-1)^2 - 4$: The curve $y = x^2$ can be changed into the curve of the function $y = (x-1)^2 - 4$ as follows:

1. By moving the Y-axis one unit to the left, we get the curve $y = (x-1)^2$.

2. By moving the X-axis four units up the Y-axis, the curve of $y = (x-1)^2$ becomes $y = (x-1)^2 - 4$.

$\therefore$ The graph of $y = ax^2 + bx + c$ where $a > 0$ is a parabola opening upward. When $a < 0$, the graph $y = ax^2 + bx + c$ opens downward.

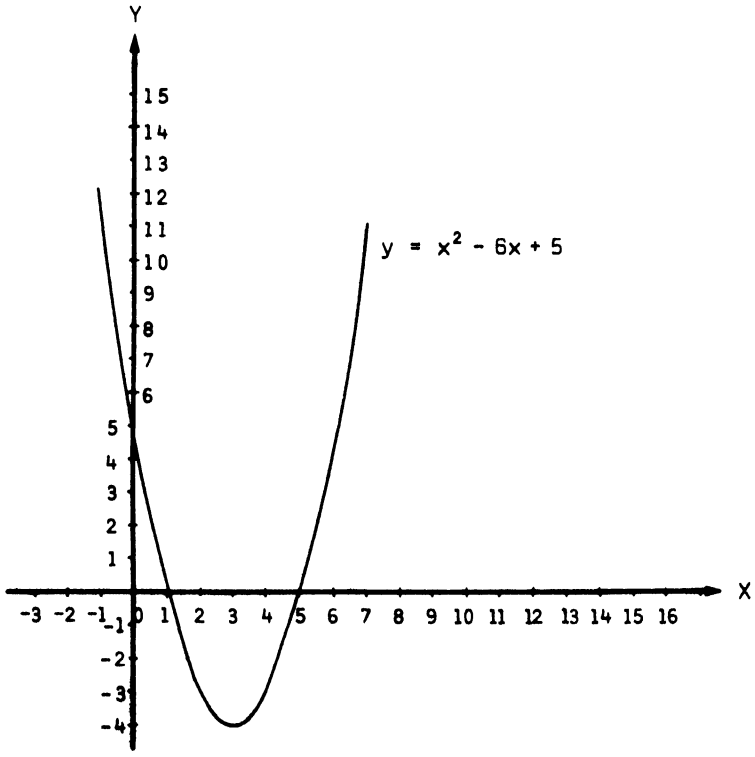
Example 1:

Plot the graph of $y = x^2 - 6x + 5$.

We write the following table of pairs:

x	-1	0	1	2	3	4	5	6	7
y	12	5	0	-3	-4	-5	0	5	12

We then draw the graph:



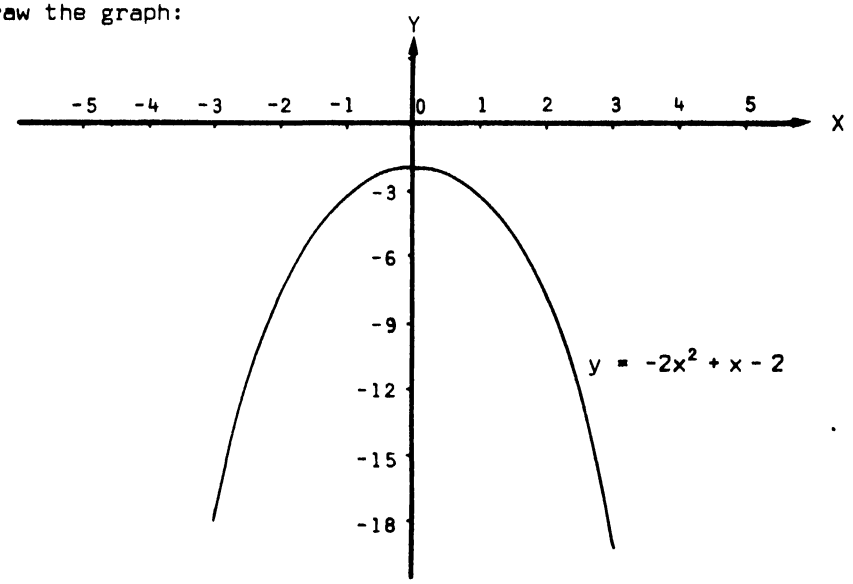
Example 2:

Plot the graph of $y = -2x^2 + x - 2$

The table of pairs is as follows:

x	-2	-1	0	1	2	3
y	-12	-5	-2	-3	-8	-17

We then draw the graph:



The graphs of the equations $ax^2 + bx + c = 0$ have the following important properties[1]:

1. If $a > 0$, the graph opens upward

2. If $a < 0$, the graph opens downward

3. If $ax^2 + bx + c = 0$ has the real solutions x_1 and x_2, where $x_1 \neq x_2$, the graph crosses the X-axis at $x = x_1$ and $x = x_2$

4. If $ax^2 + bx + c = 0$ has two equal real solutions $x_1 = x_2 = \hat{x}$, the graph is tangent to the X-axis at $x = \hat{x}$

5. If $ax^2 + bx + c = 0$ has non-real solutions, the graph does not cross the X-axis

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1/ See also Section 2.8.3

6. The graph is symmetric about the

$$x = -\frac{b}{2a}$$

7. The vertex of the parabola is the point

$$\left(-\frac{b}{2a}, \frac{-b^2+4ac}{4a}\right)$$

2.9 Quadratic equations

Plotting the graph of a quadratic function led logically to the solution of a quadratic equation, i.e. an equation of a second degree (see section above). However, the accuracy of this method is limited. We must therefore find an algebraical solution to the problem.

1. Solve the equation $x^2 = 4$

From the curve of $y = x^2$ it is found that if $x^2 = 4$ the corresponding values of x are $x_1 = +2$ and $x_2 = -2$. The reasoning may be thus stated:

$$x^2 = 4$$

$$x = \pm\sqrt{4}$$

and, $x_1 = 2$ and $x_2 = -2$

2. Solve the equation $(x-1)^2 - 4 = 0$

$$(x-1)^2 = 4$$

$$(x-1) = \pm\sqrt{4}$$

$$x = 1 \pm \sqrt{4}$$

Thus, $x_1 = 1 + 2 = 3$, $x_2 = 1 - 2 = -1$

3. Solution of the general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$

(i) $ax^2 + bx = -c$

(ii) $x^2 + \frac{b}{a}x = -\frac{c}{a}$

(iii) Adding the square of half the coefficient of x,

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

(iv) Taking square roots,

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or in full,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and,

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

If a,b,c are real, we may easily deduce the following properties of the solutions:

When $b^2 - 4ac$ is positive, the two solutions (x_1, x_2) are real and unequal.

When $b^2 - 4ac$ is zero, the two solutions are real and equal, $x_1 = x_2 = x$.

When $b^2 - 4ac$ is negative, the two solutions are unequal and neither of them is real.

If x_1 and x_2 are the roots of $ax^2 + bx + c = 0$, the following equations are valid:

$$x_1 + x_2 = -\frac{b}{a} \quad , \quad x_1 x_2 = \frac{c}{a}$$

Example 1: Solve the equation $x^2 - 4x + 3 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 3}}{2 \times 1} = \frac{4 \pm \sqrt{16 - 12}}{2}$$

$$x_1 = \frac{4 + \sqrt{4}}{2} = 3 \quad , \quad x_2 = \frac{4 - \sqrt{4}}{2} = 1$$

Example 2: Solve the equation $\frac{1}{x-1} = \frac{2}{x-3} - \frac{2}{3}$

Clearing factors by multiplying throughout by $3(x-1)(x-3)$

$$3(x-3) = 2 \times 3(x-1) - 2(x-1)(x-3)$$
$$3x - 9 = 6x - 6 - 2(x^2 - 4x + 3)$$
$$3x - 9 - 6x + 6 + 2x^2 - 8x + 6 = 0$$

Collecting like terms,

$$2x^2 - 11x + 3 = 0$$

Putting $a = 2$, $b = -11$, $c = 3$ we get:

$$x = \frac{11 \pm \sqrt{121 - 24}}{4} = \frac{11 \pm \sqrt{97}}{4} = \frac{11 \pm 9.849}{4}$$

or,

$$x_1 = \frac{11 + 9.849}{4} = 5.212 \quad , \quad x_2 = \frac{11 - 9.849}{4} = 0.288$$

Example 3: Solve the equation $x^2 + 2x + 5 = 0$

$$x = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$x_1 = \frac{-2 + 4i}{2} = -1 + 2i \quad , \quad x_2 = \frac{-2 - 4i}{2} = -1 - 2i$$

Note: $\sqrt{-1} = i$

Example 4:

The sum of two numbers is 14, and their product is 45
Write the quadratic equation of which they are the solutions, and solve.

If x_1, x_2 are the two solutions of the quadratic equation, we have:

$$ax^2 + bx + c = ax^2 - a(x_1 + x_2)x + ax_1x_2$$
$$= ax^2 - a(-\frac{b}{a})x + a(\frac{c}{a})$$

$$x_1 + x_2 = 14 \quad , \quad x_1x_2 = 45$$

Substituting, we get:

$$ax^2 - a(14)x + a45 = 0$$

or,

$$x^2 - 14x + 45 = 0$$

$$x = \frac{14 \pm \sqrt{196 - 4 \times 1 \times 45}}{2 \times 1} = \frac{14 \pm 4}{2}$$

$$x_1 = 9 \quad , \quad x_2 = 5$$

Example 5:

The distance (h) which a body reaches in time t is projected vertically upward with velocity u, given by the formula:

$$h = ut - \tfrac{1}{2} gt^2$$

If $u = 160$ and $g = 32$, find the time a body takes to rise 240 ft.

Substituting, we get:

$$240 = 160\, t - 16\, t^2$$

or,

$$16\, t^2 - 160\, t + 240 = 0$$

$$t^2 - 10\, t + 15 = 0$$

$$t = \frac{10 \pm \sqrt{100 - 60}}{2}$$

$$t_1 = \frac{10 + \sqrt{40}}{2} = 8.16$$

$$t_2 = \frac{10 - \sqrt{40}}{2} = 1.84$$

3.0 Simultaneous equations of second degree

These are equations which involve two variables which include terms of a second degree.

1. A mixed system consisting of a linear equation and a quadratic equation:

$$y = ax + b$$
$$y = ax^2 + bx + c$$

The method of the solution is quite straightforward. Set the two right-hand sides equal and solve the resulting quadratic equation for x. Then substitute the solution(s) into the linear equation and find the corresponding values for y:

Example: Solve

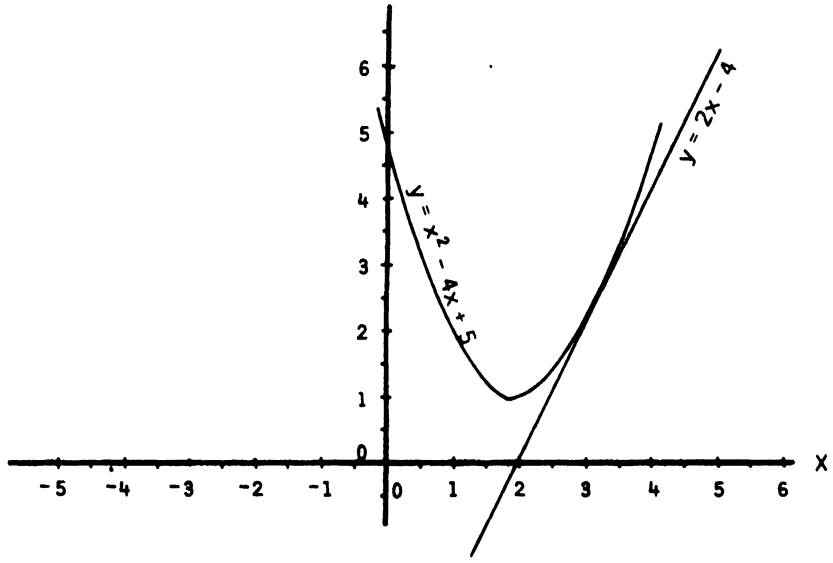
$$y = 2x - 4$$
$$y = x^2 - 4x + 5$$

$$2x - 4 = x^2 - 4x + 5$$
$$x^2 - 6x + 9 = 0$$

$$x = \frac{6 \pm \sqrt{36 - 36}}{2}$$
$$x_1 = x_2 = \mathring{x} = 3$$

Substituting, for $\mathring{x} = 3$, $y = 2$

The graph of the straight line and parabola is given below.



The line is tangent to the parabola.

2. Symmetry equations:

Example: Solve $2x - y = 2$

$xy = 4$

This is equivalent to,

$y = 2x - 2$

$y = \frac{4}{x}$

So we solve,

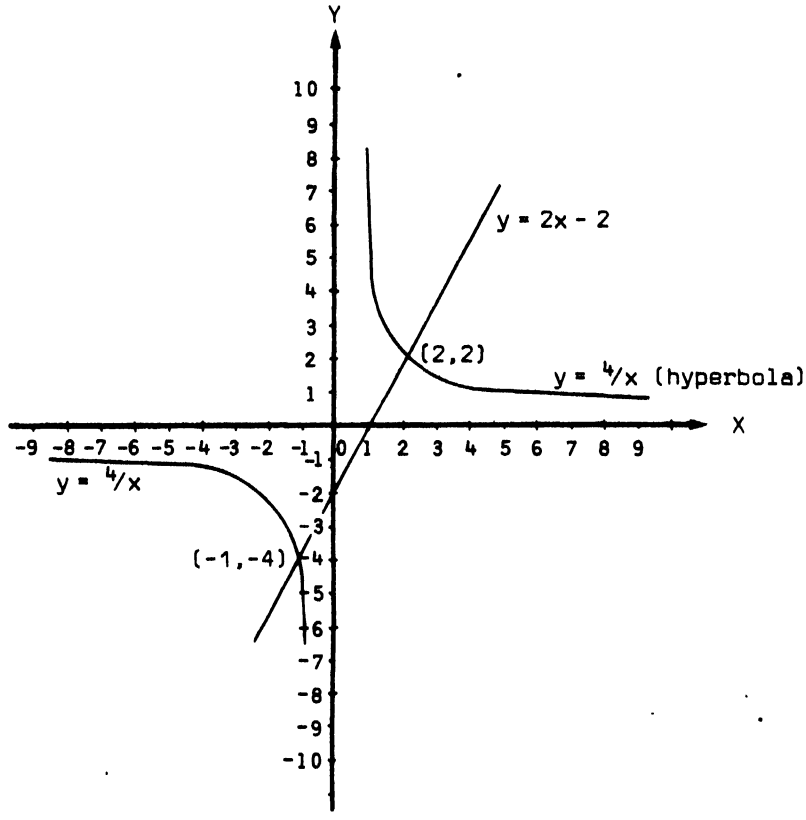
$$2x - 2 = \frac{4}{x}$$

$$2x^2 - 2x - 4 = 0 \quad , \quad x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4(-2)}}{2} = \frac{1 \pm 3}{2}$$

$$x_1 = 2 \quad , \quad x_2 = -1$$

The corresponding values of y are 2 and -4. Hence, the solutions are (2,2) and (-1, -4).



3. Both equations are of a second degree:

$$(1) \qquad x^2 + y^2 = 89$$

$$(2) \qquad xy = 40$$

The aim in this case is to obtain both $x + y$ and $x - y$.
From (2), we have:

$$(3) \qquad 2xy = 80$$

Adding (3) to (1) we get:

$$x^2 + 2xy + y^2 = 169$$

$$(x + y)^2 = 169$$

$$(4) \qquad x + y = \pm 13$$

Subtracting (3) from (1), we get:

$$x^2 - 2xy + y^2 = 9$$
$$(x - y)^2 = 9$$
$$(5) \qquad x - y = \pm 3$$

Associating the equations (4) and (5) we get:

$$x + y = \pm 13$$
$$x - y = \pm 3$$

$x = +8 \quad y = +5$ $\qquad$ $x = 5 \quad y = +8$

$x = -8 \quad y = -5$ $\qquad$ $x = -5 \quad y = -8$

3.1 The rational quadratic function

The most general rational quadratic function is of the form:

$$\frac{ax^2 + bx + c}{a_1x^2 + b_1x + c_1}$$

where a, b, c, a_1, b_1, c_1 are constants.

Example:

Find the minimum and maximum values of the function:

$$\frac{x^2 - 3x + 1}{2x^2 - 3x + 2}$$

for real values of x.

Let:
$$\frac{x^2 - 3x + 1}{2x^2 - 3x + 2} = y$$

$$x^2 - 3x + 1 = y(2x^2 - 3x + 2)$$

Arranging the equation as a quadratic in x, we have:

$$x^2(1 - 2y) + x(3y - 3) + (1 - 2y) = 0$$

Now for x to be real, the roots of this quadratic are real, and hence:

$$b^2 - 4ac \geqslant 0$$

Substituting, we get:

$$(3y - 3)^2 - 4 \times (1 - 2y)(1 - 2y) \geqslant 0$$

$$9y^2 - 18y + 9 - 4 - 16y - 16y^2 \geqslant 0$$

$$-7y^2 - 2y + 5 \geqslant 0$$

$$0 \geqslant 7y^2 + 2y - 5$$

$$0 \geqslant (7y - 5)(y + 1)$$

So for x to be real, $(7y - 5)(y + 1)$ must be negative.
That is, y must lie between the values $\frac{5}{7}$ and -1:

$$-1 \leqslant y < \frac{5}{7}$$

The maximum value of the function $= \frac{5}{7}$

The minimum value of the function $= -1$

3.2 Graphical solutions of equations

Find, graphically, the real roots of the equation: $x^3 + 4x - 8 = 0$

The solution can be effected in two ways:

(a) By drawing the graph of $y = x^3 + 4x - 8$ and finding the values of x, for which y is zero

(b) By drawing the graphs of $y = x^3$ and $y = 8 - 4x$ and finding the abscissae of the common points. For the common points,

$$\left.\begin{aligned} y &= x^3 \\ \text{and} \quad y &= 8 - 4x \end{aligned}\right\} \quad \text{and hence} \quad x^3 = 8 - 4x$$

$$\text{or} \quad x^3 + 4x - 8 = 0$$

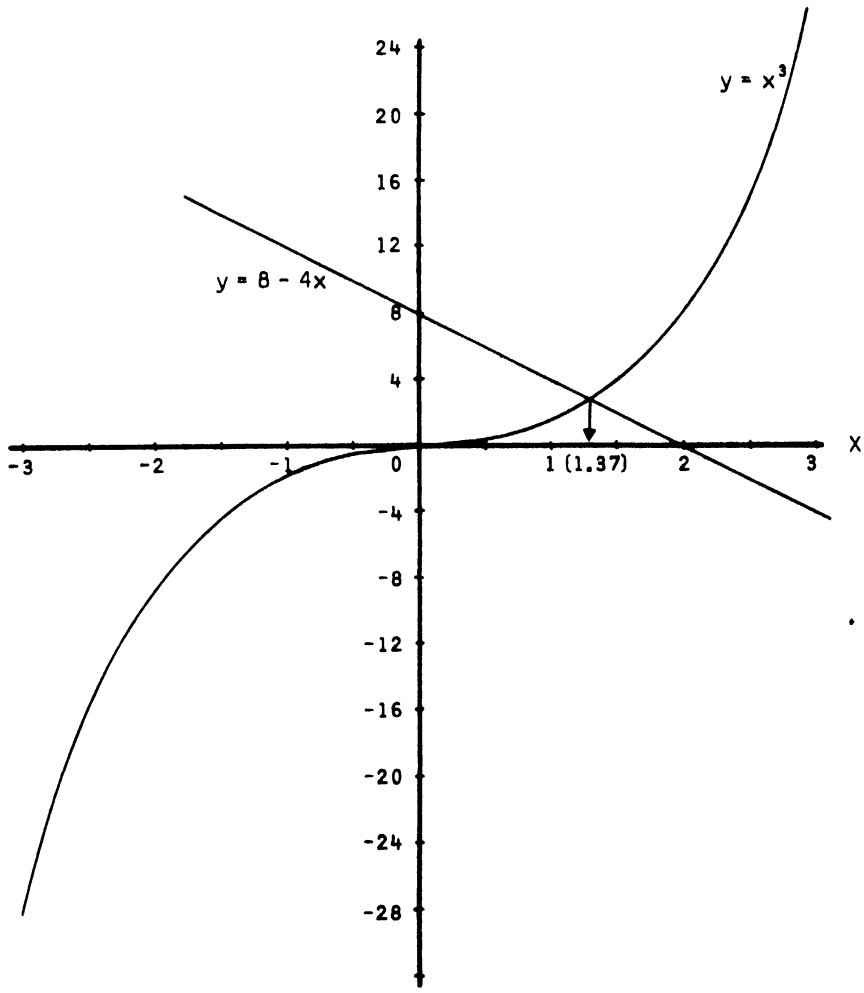
The second method will be used owing to the case with which the curve $y = x^3$ and the straight line $y = 8 - 4x$ can be drawn.

$y = x^3$

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	-27	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8	27

$y = 8 - 4x$

x	0	2
y	8	0



From the graphs $x^3 = 8 - 4x$, i.e. $x^3 + 4x - 8 = 0$, when $x = 1.37$.
As the straight line meets the curve in one point, the equation has only one real root.

3.3 Exponential functions

The function f defined by $y = a^x$ $(a > 0)$ is called the exponential function with base a. Its domain of definition is the set of real numbers. Its range of values is $0 < y < \infty$, if $a \neq 1$.

Properties:

1. For $a > 0$ and $b > 0$ and x and y real

   a) $a^x \times a^y = a^{x+y}$

   b) $(a^x)^y = a^{xy}$

   c) $(ab)^x = a^x b^x$

2. a) $a^x > 1$ for $a > 1$, x real and >0

   b) $a^x = 1$ for $a = 1$, x real and >0

   c) $a^x < 1$ for $0<a<1$, x real and >0

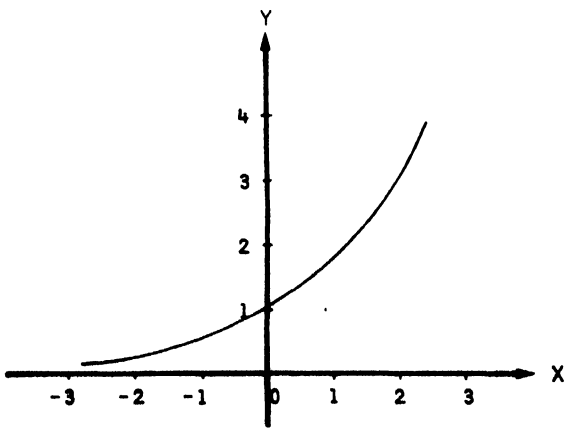
3. Let x and y be real numbers, such that $x < y$.
   We then have,

   a) $a^x < a^y$ for $a > 1$

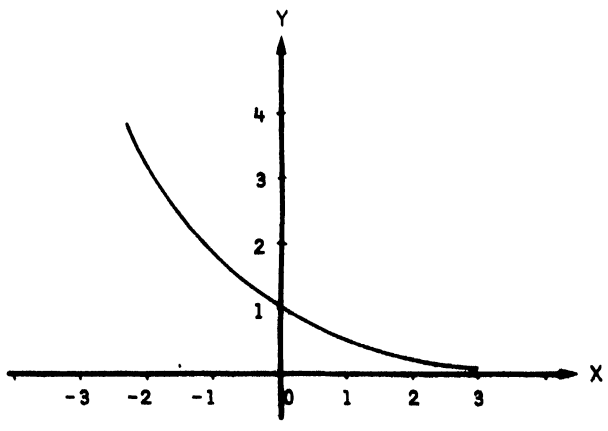
   b) $a^x = a^y$ for $a = 1$

   c) $a^x > a^y$ for $0<a<1$

Typical graphs of a^x for $a > 1$ and $a < 1$ are given below.



a^x is strictly monotone increasing when $a > 1$



a^x is strictly monotone decreasing when $a < 1$

The number $e \approx 2.71828$

There is a particular value $a > 0$ of the exponential function $y = a^x$ which is of great importance in mathematics; it is an irrational number and approximately equal: $e \approx 2.71828 \ldots$

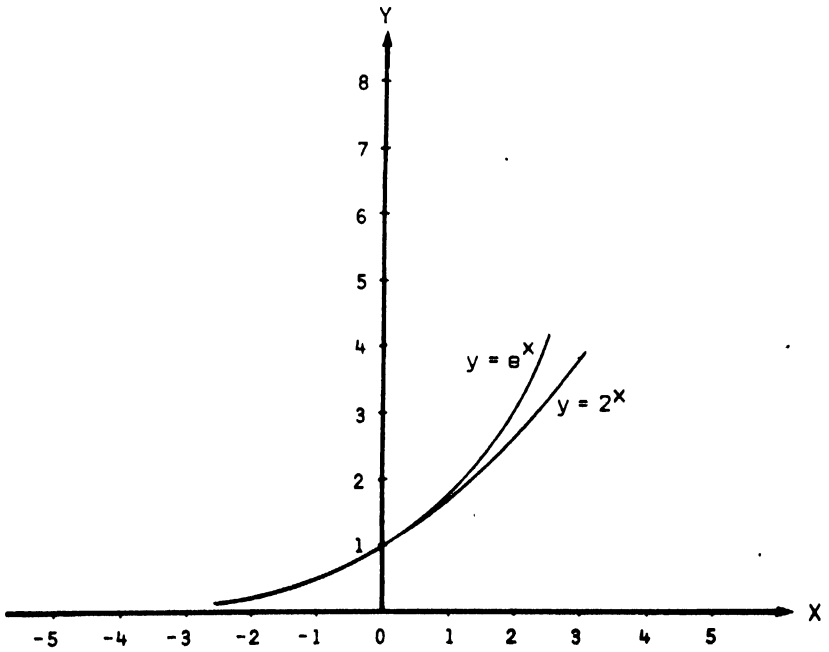
The associated exponential function $y = e^x$ is of much importance. Sometimes this function is written $y = \exp x$.

A convenient approximation to e^x is given by the polynomial:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

By definition, $n! = 1 \times 2 \times 3 \times \ldots (n-1) \times (n)$ for n a positive integer, and $0! = 1$

The graphs of the exponential functions $y = e^x$ and $y = 2^x$ are given below. Note that $y = e^x$, $y = 2^x$, and $y = a^x$ all pass through $(0,1)$. Of these $y = e^x$ is the one whose tangent 1/ at $(0,1)$ makes an angle of $45°$ with the Y-axis.



1/ **See also:** Part 3 - Trigonometry

The exponential function is also used in the definitions of some other functions which occur in mathematics. These are also called the hyperbolic functions 1/. Their definitions are:

$$\sin h\ x = \frac{e^x - e^{-x}}{2}$$

$$\cos h\ x = \frac{e^x + e^{-x}}{2}$$

$$\tan h\ x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

The function $\sin h\ x$ is read "the hyperbolic sine of x"; $\cos h\ x$, "the hyperbolic cosine of x"; and $\tan h\ x$, "the hyperbolic tangent of x".

3.4 Logarithmic functions 2/

The function inverse to that given by $y = a^x$, $a > 0$, $a \neq 1$, is written: $y = \log_a x$ and is called the logarithmic of x to the base a.

For computational purposes the base is usually taken to base 10: $y = \log_{10} x$.

For theoretical purposes the base is taken to be e(=2.7182) : $y = \log_e x$.

Frequently, the notation $\ln x$ is used for $\log_e x$.
Logarithms to base 10 are called "common" logarithms.
Those to the base e are called "natural" logarithms.

Relations:

1) $\log_a a^x = x \log_a a = x$

2) $a^{\log_a x} = x$

Properties:

1) $\log_a xy = \log_a x + \log_a y$

2) $\log_a \frac{1}{x} = -\log_a x$

3) $\log_a \frac{y}{x} = \log_a y - \log_a x$

4) $\log_a (x^y) = y \log_a x$

1/ See also Part 3 - Trigonometry

2/ See also Section 1.1.6

Note:

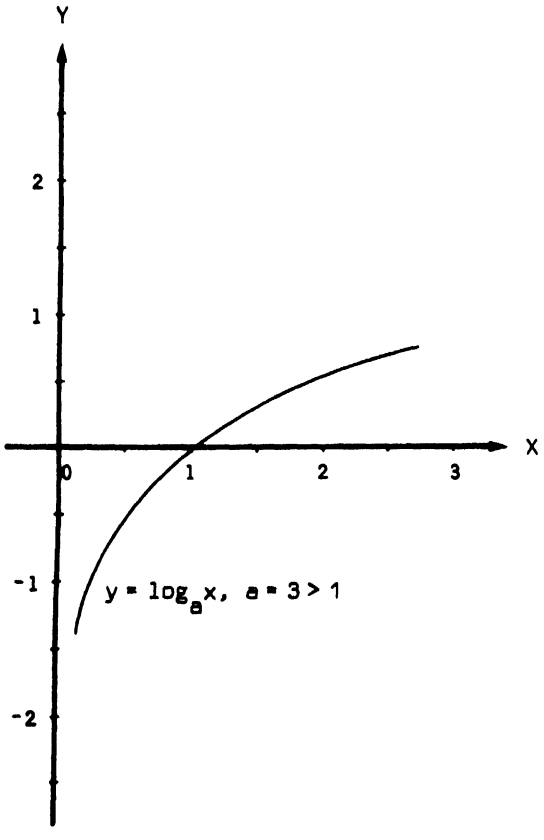
1) $$\log_b a = \frac{1}{\log_a b}$$

2) $$\log_a x = \frac{\log_b x}{\log_b a}$$

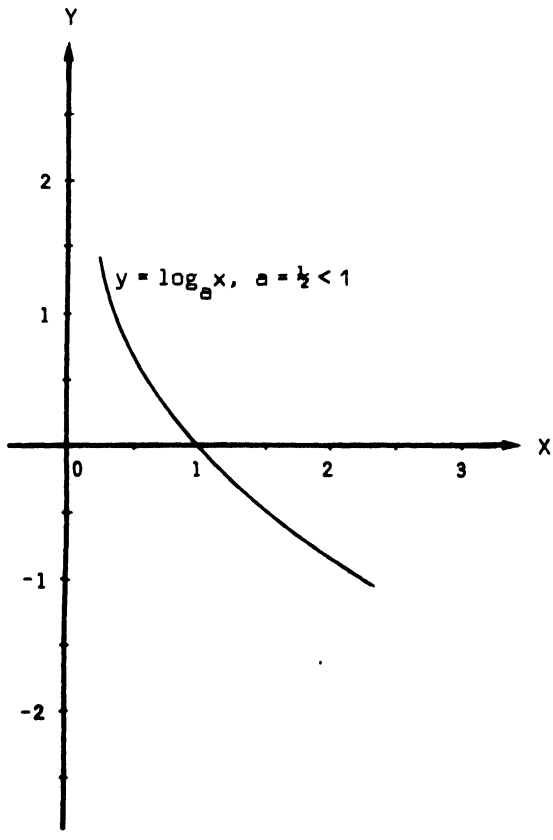
A special case of this is:

$$\log_a x = \frac{\log_e x}{\log_e a}$$

The graphs of $y = \log_3 x$ and $y = \log_{\frac{1}{2}} x$ are given below.



The graph of $y = \log_a x$ is strictly monotone increasing when $a > 1$



The graph of $y = \log_a x$ is strictly monotone decreasing when $a < 1$

3.5 Forms of inverse variation

(1) Inverse variation $y = \frac{k}{x}$ or $x = \frac{k}{y}$:

Let x and y be two quanties such that their product is constant,

$$xy = k$$

Then,

$$x = \frac{k}{y} \quad \text{or} \quad y = \frac{k}{x}$$

Each quantity can be expressed in terms of the reciprocal or inverse of the other.

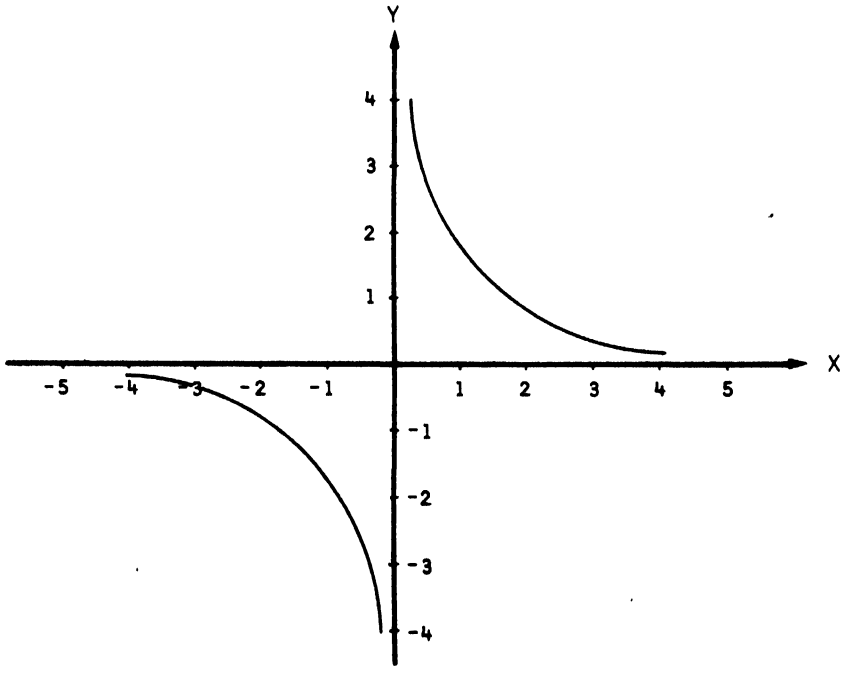
Graph of $y = \frac{k}{x}$: In its simplest form when $k = 1$, the equation becomes $y = \frac{1}{x}$. The following is a table of values formed in the usual way:

$y = 1/x$

x	$1/4$	$1/2$	1	2	3	4
y	4	2	1	$1/2$	$1/3$	$1/4$

A similar set of values can be tabulated for negative values of x, the corresponding values of y being negative.

The curve which is called a "hyperbola" is shown in the graph below.



It consists of two branches alike in shape, one for +ve values and the other for -ve values of x. The following important features of this curve should be noted:

(a) As x increases, y or $\frac{1}{x}$ decreases. When x becomes very great, y becomes very small and the curve approaches the axis of X (called the horizontal asymptote of the hyperbola).

(b) As x decreases, y or $\frac{1}{x}$ increases. When x is very small, y is very large and the curve approaches very close to the axis of Y (called the vertical asymptote of the hyperbola).

Both these properties of the curve are repeated for -ve values of x.

(2) Other forms of inverse variation:

(i) One quantity may vary inversely as the square of another quantity,

$$y = \frac{1}{x^2} \quad \text{and} \quad y = \frac{k}{x^2}$$

(ii) Another form of variation is that in which one quantity varies inversely as the cube of another,

$$y = \frac{1}{x^3} \quad \text{and} \quad y = \frac{k}{x^3}$$

(iii) In general, y may vary directly or inversely as any power of x,

$$y = x^n \quad \text{or} \quad y = \frac{1}{x^n} \quad \text{and} \quad y = \frac{k}{x^n}$$

3.6 Series: The Arithmetic, Geometric and Harmonic Series - The Binomial Series -

Series:

A series is a sequence of terms, each of which can be obtained by some definite law.

Examples of series are:

(i) 1, 2, 3, 4, 5

(ii) 1, 4, 16, 64, ...

(iii) $1^2, 2^2, 3^2, 4^2, \ldots$

(iv) $\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \ldots$

(v) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}, \ldots$

3.6.1 The Arithmetic Series or Arithmetic Progression (A.P.)

An Arithmetic Series is one in which each term is formed from that immediately preceding it by adding or subtracting a constant number. The number thus added or subtracted is called the "common difference" of the series.

(1) 7, 10, 13, 16, ... (common difference, d = 3)

(2) 8, 6, 4, 2, 0, -2, ... (d = -2)

In general, if three numbers a,b,c are in Arithmetic Progression, then:

$$b - a = c - b$$

The general term (a_n) and the sum (S_n) of an Arithmetic Series:

1. If a is the first term of a series, and d the common difference, the series can be written:

   $$a, a + d, a + 2d, a + 3d, \ldots$$

   Thus, the fourth term is:

   $$a + (n - 1)d = a + 3d$$

Hence, if the number of any term is denoted by the n^{th} term, we have:

$$n^{th} = a_n = a + (n - 1)d$$

The sum of any number of terms of an Arithmetic Progression is given by:

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right]$$

or,

$$S_n = \frac{n(a + a_n)}{2}$$

Arithmetic mean: If three numbers are in an Arithmetic Progression, the middle one is called the arithmetic mean of the other two.

Let a,b,c be the three terms:

$$b - a = c - b \quad \text{and} \quad 2b = a + c, \quad \text{and} \quad b = \frac{a + c}{2}$$

Exercise 1:

Find the 21^{st} term and the sum to 24 terms of the series 2, 4, 6, 8, ...

Solution: The series is an Arithmetic Progression:

$$a = 2 \quad \text{and} \quad d = 2$$

$$a_{21} = 2 + (21 - 1) \times 2 = 42$$

and

$$S_{24} = \frac{24}{2}\left[2 \times 2 + (24 - 1)2\right] = 600$$

Exercise 2:

The sum of an Arithmetic Progression of 8 terms is 90 and a = 6.
Find the value of d.

Solution:
$$90 = \frac{8}{2}\left[2 \times 6 + (8 - 1)d\right]$$

$$90 = 4(12 + 7d)$$

$$90 = 48 + 28d$$
$$28d = 42$$
$$d = 1.5$$

Exercise 3:

How many terms of the series 3, 6, 9, ... must be taken so that their sum is 135?

$$135 = \frac{n}{2}\left[2 \times 3 + (n - 1)3\right]$$

$$270 = n(6 + 3n - 3)$$

$$270 = n(3 + 3n)$$

$$270 = 3n^2 + 3n$$

$$3n^2 + 3n - 270 = 0$$

$$n^2 + n - 90 = 0$$

Factorising,

$$(n - 9)(n + 10) = 0$$

$$n = 9 \quad \text{or} \quad -10$$

The root -10 is inadmissible as having no meaning in this connection.

The solution is: $n = 9$.

3.6.2 Geometric Series or Geometric Progression (G.P.)

The geometric progression is a series of terms in which each term is obtained from the preceding one by multiplying by a constant quantity, called the common (or constant) ratio (r).

The following series are geometric progressions:

(i) 1, 2, 4, 8, 16, ... (common ratio, $r = 2$)

(ii) $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{18}$, ... (common ratio, $r = \frac{1}{3}$)

(iii) 2, -8, 32, -128, ... (common ratio, $r = -4$)

The general term (a_n) and sum of n terms (S_n) of a Geometric Series

The most general geometric series is of the form:

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \ldots$$

The n^{th} term of the series is given by:

$$a_n = ar^{n-1}$$

The sum of the n terms is obtained by:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Geometric mean: The geometric mean of two numbers a and b is x, if a,x,b are consecutive terms in a geometric progression,

we have,

$$\frac{x}{a} = \frac{b}{x} = r$$

$$x^2 = ab$$

$$x = \sqrt{ab}$$

Exercise:

Find the 5^{th} term and the sum of the seven terms of the series 2, 3, 4.5, ...

Solution: $r = \frac{b}{a} = \frac{3}{2} = 1.5$

(1) $a_5 = 2 \times 1.5^4$

$\log_{10} a_5 = \log_{10} 2 + 4 \log 1.5$

$\log_{10} a_5 = 0.3010 + 4 \times 0.1761$

$\log_{10} a_5 = 1.0054$

$a_5 = 10.125$

(2) $$S_7 = \frac{2\,(1 - 1.5^7)}{1 - 1.5} = \frac{2\,(1 - 17.085)}{1 - 1.5}$$

$$S_7 = \frac{2 \times (-16.085)}{-0.5} = 64.35$$

3.6.2.1 Infinite Geometric Series

Increasing Series: If the common ratio of a geometric series is numerically greater than unity $(r > 1)$, as n the number of terms approaches infinity, S_n, the sum of these terms also approaches infinity. This may be expressed by the following notation:

$$n \to \infty, \text{ then } S_n \to \infty$$

Decreasing Series: If the common ratio of the series is numerically less than unity $(r < 1)$, as n increases without limit, the terms themselves decrease without limit and ultimately become indefinitely small.

The above suggests the general treatment of this question, using:

$$S_n = \frac{a\,(1 - r^n)}{1 - r} = \frac{a}{1 - r} - a\frac{r^n}{1 - r}$$

As, $$n \to \infty, \quad r^n \to 0, \quad a\frac{r^n}{1 - r} \to 0$$

Thus, the right hand side approaches $\frac{a}{1 - r}$ as a limit.

This is the "limiting sum" of a diminishing Geometric Progression and is called the "sum of infinity". If it is presented by S_∞, then:

$$S_\infty = \frac{a}{1 - r}$$

Example:

Find the sum of infinity of the series: $2, \frac{1}{2}, \frac{1}{8}, \ldots$

$$r = \frac{b}{a} = \frac{1}{2} \div \frac{2}{1} = \frac{1}{4}$$

$$S_\infty = \frac{2}{1 - 1/4} = \frac{2}{3/4} = 8/3 = 2\,2/3$$

3.6.3 Harmonic Series or Harmonic Progression (H.P.)

A series of numbers is said to be in Harmonic Progression (H.P.) if their reciprocals form a series in Arithmetic Progression.

Thus the series $1, \quad 4, \quad 7, \quad 10, \quad \ldots,$ are in A.P.

Thus the series $1, \quad \frac{1}{4}, \quad \frac{1}{7}, \quad \frac{1}{10}, \quad \ldots,$ are in H.P.

Harmonic mean: Let a and b be two numbers of H.P.
Let H be their harmonic mean.

$$a, \quad H, \quad b, \qquad \text{are in H.P.}$$

then, $$\frac{1}{a}, \quad \frac{1}{H}, \quad \frac{1}{b}, \qquad \text{are in H.P.}$$

$$\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$\frac{1}{H} = \frac{a + b}{2ab}$$

$$H = \frac{2ab}{a + b}$$

3.6.4 Binomial Series

Let n be a positive integer. Then 1/,

(1) $$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + b^n$$

or in the expression $(a+b)^n$, the coefficient of $a^{n-r}b^r$ is $\binom{n}{r}$.

The binomial series permits us to write down rather quickly the expansions of powers of binomials which are tedious to compute by repeated multiplication.

1/ See also Section 1.6.2

Example 1:

Expand $(2x + 5y)^3$:

$$(2x + 5y)^3 = (2x)^3 + \binom{n}{1}(2x)^2(5y) + \binom{n}{2}(2x)(5y)^2 + (5y)^3$$

$$= 8x^3 + 3 \times 4x^2 \times 5y + 3 \times 2x \times 25y^2 + 125y^3$$

$$= 8x^3 + 60x^2 + 150xy^2 + 125y^3$$

Example 2:

Expand $(3x - 2y)^4$:

$$(3x - 2y)^4 = (3x)^4 - \binom{4}{1}(3x)^3(2y) + \binom{4}{2}(3x)^2(2y)^2 - \binom{4}{3}3x(2y)^3 + (2y)^4$$

$$= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4 \quad .$$

Another expression of Binomial Series:

According to the expression (1) above, we have,

$$(x + a)^n = x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \ldots + \binom{n}{n-1}xa^{n-1} + a^n$$

or,

$$(x + a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}a^3 + \ldots + a^n$$

Every binomial expression can be reduced to one of these forms, as follows:

$$(x + a)^n = \left[x\left(1 + \frac{a}{x}\right)\right]^n$$

$$= x^n\left[1 + n\left(\frac{a}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^3 + \ldots + \left(\frac{a}{x}\right)^n\right]$$

Similarly,

$$(x - a)^n = x^n\left(1 - \frac{a}{x}\right)^n$$

It has been proved that the binomial expression is held for all values of n.
For example, $n = \frac{1}{2}$

$$(1 + x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \ldots$$

Note:

If n is a positive integer, the number of terms of $(1 + x)^n$ is finite and will clearly be $n + 1$. Consequently, if a value is assigned to x the sum of the series will be a finite number and the series is said to be convergent.

If n is fractional or negative, the number of terms is infinite. If the sum increases without limit as n increases, the series is said to be divergent. But it can be shown that if x, in $(1 + x)^n$ is numerically less than unity, the sum of the series will approach a limit, as in the case with certain geometric series. Subject to this condition, the series is always convergent.

Examples:

Find the given term in the following expansions:

1. x^2 in expansion of $(1 + x)^8$:

$$\binom{8}{2} 1^6 x^2 = \frac{8!}{2!6!} x^2 = 28x^2$$

2. x^3 in expansion of $(3 - 2x)^4$:

$$-\binom{4}{3} 3^1 (2x)^3 = -\frac{4!}{1!3!} 3 \times 8x^3 = -96x^3$$

3. x^4 in expansion of $(x^2 + \frac{2}{\cdot})^5$:

$$(x^2 + \frac{2}{x})^5 \qquad \left[x^2 (1 + \frac{2}{x^3})\right]^5 = x^{10} (1 + \frac{2}{x^3})^5 ,$$

$$x^{10} \left[\binom{5}{2} 1^3 (\frac{2}{x^3})^2\right] = x^{10} \times \frac{5!}{2!3!} \times \frac{4}{x^6} = 40x^4$$

4. Constant term in expansion of $(2x + \frac{1}{x^2})^3$:

$$(2x + \frac{1}{x^2})^3 = \left[2x (1 + \frac{1}{2x^3})\right]^3 = 8x^3 (1 + \frac{1}{2x^3})^3 ,$$

$$8x^3 \left[\binom{n}{1} (\frac{1}{2x^3})^1\right] = 8x^3 \times \frac{3!}{1!2!} \times \frac{1}{2x^3} = 12$$

5. Constant term in expansion of $(x^2 - \frac{1}{x})^6$:

$$(x^2 - \frac{1}{x})^6 = \left[x^2 (1 - \frac{1}{x^3})\right]^6 = x^{12} (1 - \frac{1}{x^3})^6 ,$$

$$x^{12} \left[\binom{n}{4} (\frac{1}{x^3})^4\right] = x^{12} \times \frac{6!}{4!2!} \times \frac{1}{x^{12}} = 15$$

3.7 Logarithms

Definition of logarithms:

The logarithm of a number to a given base, assumed positive, is the index of the power to which the base must be raised in order to equal the number, e.g.

The logarithm of 16 to base 2 is 4, because $16 = 2^4$

The logarithm of $1/9$ to base 3 is -2, because $1/9 = 3^{-2}$

The logarithm of a number A to base a is written: $\log_a A = x$

Properties of logarithms:

1. If $\log_a A = x$, then $A = a^x$

2. $\log_a A = x$ and $\log_a B = y$

   $A = a^x$ and $B = a^y$

   $AB = (a^x)(a^y) = a^{x+y}$

3. $\log_a (AB) = \log_a A + \log_a B$

4. $\log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B$

5. $\log_a A = x$ and $\log_a B = y$

   $\frac{A}{B} = \frac{a^x}{a^y} = a^{x-y}$

6. $\log_a A^\mu = \mu \log_a A$

7. $\log_a A = x$, $A = a^x$

   $A^\mu = (a^x)^\mu = a^{x\mu}$

   $\log_a A^\mu = \log_a (a^{x\mu}) = x\mu \log_a a = x\mu$

8. $\log_a A = x$, $A = a^x$

   $\frac{1}{A} = \frac{1}{a^x} = a^{-x}$

   $\log_a \frac{1}{A} = -x \log_a a = -x$

9. $\log_a \frac{1}{A} = \log_a A^{-1} = -\log_a A$

Exercises:

1. Given $\log_{10}2 = 0.30103$, $\log_{10}3 = 0.47712$

find $\log_{10}8$, $\log_{10}\frac{27}{4}$ and $\log_{10}900$.

(i) $\log_{10}8 = \log_{10}2^3 = 3\log_{10}2 = 3 \times 0.30103 = 0.90309$

(ii) $\log_{10}(\frac{27}{4}) = \log_{10}27 - \log_{10}4 = \log_{10}3^3 - \log 2^2$

$= 3\log_{10}3 - 2\log_{10}2 = 0.82930$

(iii) $\log_{10}900 = \log_{10}(9 \times 10^2) = \log_{10}3^2 + 2\log_{10}10$

$= 2\log_{10}3 + 2\log_{10}10 = 2.95424$

2. Simplify $\log \sqrt{3} \div \log 27$.

$$\log 3^{\frac{1}{2}} \div \log 3^3 = \frac{\frac{1}{2}\log 3}{3\log 3} = \frac{1}{6}$$

3. If $y = 10\log_{10}\overline{\rho}_v$ find the antilog of y.

$$\text{ant} - y = \overline{\rho}_v^{10}$$

4. If $10\log_{10}\overline{C} = 10\log_{10}3430 + S_v - TS_{1\ kg}$

find the function of $\overline{C}$.

$$\log_{10}\overline{C} = \log_{10}3430 + \frac{1}{10}(S_v - TS_{1\ kg})$$

$$\overline{C} = 3430 \times 10^{0.1(S_v - TS_{1\ kg})}$$

5. Noting that $\log_a A = \frac{\log_b A}{\log_b a}$

find $\log_e 500$ by using logarithms of base 10.

$$\log_e 500 = \frac{\log_{10}500}{\log_{10}e} = 6.2146$$

6. Find $\log_3 4 = \frac{\log_{10}4}{\log_{10}3} = 1.26$

The characters of a logarithm:

The integral or whole number of a logarithm is called the characteristic and the decimal part is called the mantissa.

The characteristic of a logarithm can also be calculated by inspection when logarithms are calculated to base 10, as will be seen from the following considerations:

10^4	=	10000	$\log_{10}10000$	=	4
10^3	=	1000	$\log_{10}1000$	=	3
10^2	=	100	$\log_{10}100$	=	2
10^1	=	10	$\log_{10}10$	=	1
10^0	=	1	$\log_{10}1$	=	0
10^{-1}	=	0.1	$\log_{10}0.1$	=	-1
10^{-2}	=	0.01	$\log_{10}0.01$	=	-2
10^{-3}	=	0.001	$\log_{10}0.001$	=	-3

and so on

From these results we see that:

for numbers between	1000 and 10000	the characteristic is	3
for numbers between	100 and 1000	the characteristic is	2
for numbers between	10 and 100	the characteristic is	1

and so on

It is evident that the characteristic is always one less than the number of digits in the whole number part of the number.

The mantissa alone is given in the tables (see Appendix)

Thus, $\log_{10}31 = 1.4914$, where 1 is the characteristic of the logarithm and $.4914^{10}$ is the mantissa given by the tables.

It should be noted that the mantissa is always a positive number, although the characteristic of a logarithm might be a positive or negative number:

$$\log_{10}0.31 = (\bar{1}).(4914)^{+}$$

$$= -0.5086 \text{ , algebraic number}$$

Anti-logarithms:

The anti-logarithm is a number which corresponds with a given logarithm. For this purpose, the table of logarithms is usually used:

1) The mantissa of the log only is used in the table.

2) When the significant figures of the number have been obtained, the next step is to fix the decimal point in them by using the rules which we have considered for the characteristic.

Example 1:

Find the number whose logarithm is 2.3714 :

a) Using the mantissa, .3714, we find the anti-logarithm as 2352

b) Since the characteristic is 2 the number is 235.2

Example 2:

Find the $\log_{10} 0.3185$:

$\log_{10} 0.3185 = \bar{1}.5031 = -0.4969$

Example 3:

Find the number whose $\log_{10}$ is $\bar{3}.5416$:

a) Using the mantissa, .5416, we find the anti-logarithm as 3480

b) As the characteristic is $\bar{3}$ there will be two zeros after the decimal point. The number is 0.003480

Operations with logarithms:

1. Find the sum of the logarithms 2.3415 , 0.7374 , 1.6173 :

$$\begin{array}{r} 2.3415 \\ 0.7374 \quad (+) \\ 1.6173 \\ \hline 4.6962 \end{array}$$

2. Find the sum of the logarithms $\bar{1}.6173$, $\bar{2}.3415$, $\bar{1}.6493$, 0.7374 :
Arranging thus,

$$\begin{array}{r} \bar{1}.6173 \\ \bar{2}.3415 \\ \bar{1}.6493 \quad (+) \\ 0.7374 \\ \hline \bar{2}.3455 \end{array}$$

Remember, $\bar{2}.3455 = (-1-2-1) + (0.6173 + 0.3415 + 0.6493 + 0.7374)$.

3. From the logarithm $\bar{1}.6175$ subtract the log $\bar{3}.8463$.
The log numbers can be expressed algebraically as:

log		Alg	
$\bar{1}.6175$	=	-0.3825	(-)
$\bar{3}.8463$	=	-2.1537	
		1.7712	(log = alg. number)

4. Multiply $\bar{2}.8763$ by 3:

$$\log \bar{2}.8763 = -1.1237 \quad \text{alg. number}$$

$$3 \times \bar{2}.8763 = 3 \times (-1.1237) = -3.3711 \quad \text{alg. number}$$

The algebraic number -3.3711 can be converted into a log number by subtracting 1 from the integer part of the number, and adding +1 to the decimal part:

$$(-3-1).(1-0.3711) = \bar{4}.6289$$

3.8 Permutations and Combinations [1/]

Permutations

The number of permutations of n different objects is:

$$n\,(n-1)\,(n-2)\cdots 3 \times 2 \times 1 = n!$$

Example:

The number of ways of arranging 6 different things in order is found by choosing one of the 6 for the first position (6 ways); then one of the remaining 5 for the next position (5 ways); then one of the remaining 4 for the third position (4 ways), etc.
So the total number of permutations is:

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$$

If we now ask how many ways we can arrange (or permute) 4 out of the group of 6 things, which we shall call the number of permutations of 4 out of 6, and write ${}^{6}P_{4}$, we see that:

$${}^{6}P_{4} = 6 \times 5 \times 4 \times 3 = 360$$

or,

$${}^{6}P_{4} = \frac{6!}{(6-4)!} = \frac{6!}{2!}$$

In general,

$${}^{n}P_{r} = n\,(n-1)\,(n-2)\cdots(n-r+1)$$

$$\frac{n!}{(n-r)!}$$

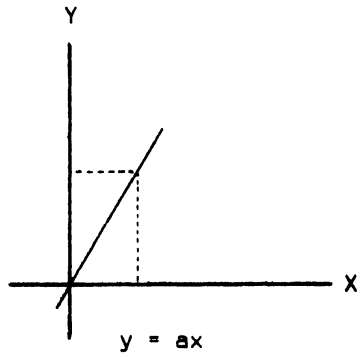
Combinations

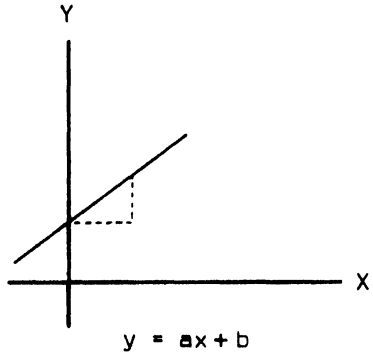
If we have a set of n different objects, in how many ways can we choose (irrespective of order) r out of them? We shall call this the number of combinations of r objects out of n written,

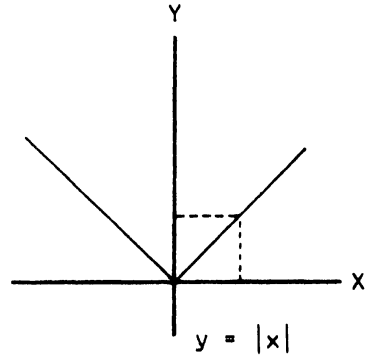
$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

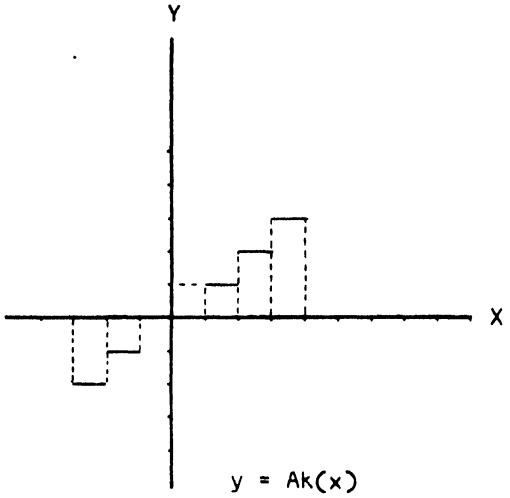
1/ See: Bazigos, G.P. (1974) - Applied Fishery Statistics, FAO-FIPS/T135

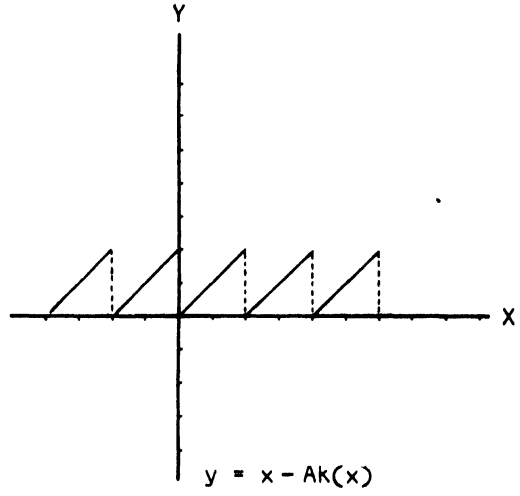
Curves illustrating a number of different types of mathematical functions

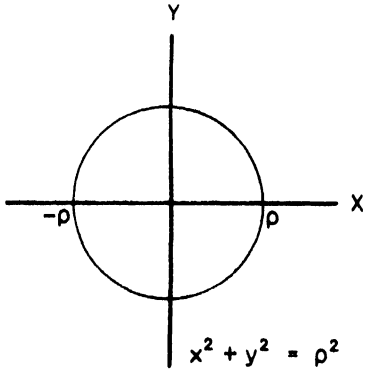


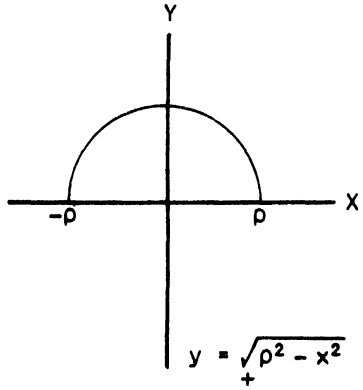


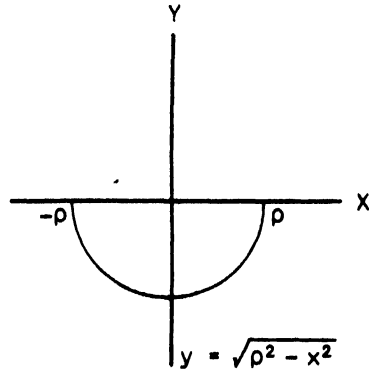


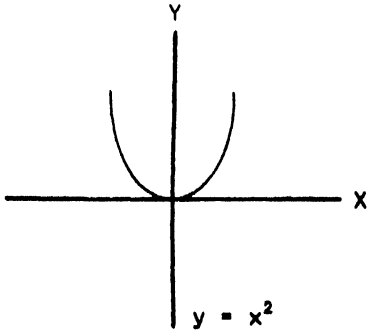


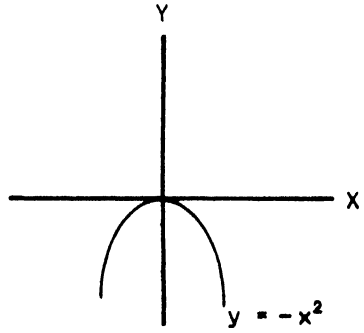


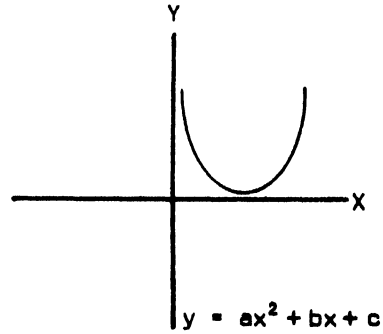


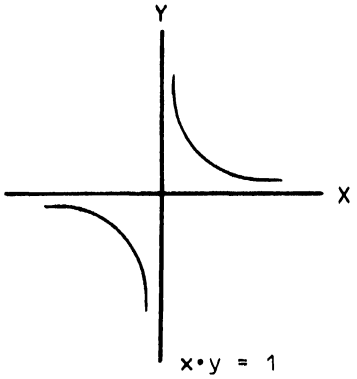




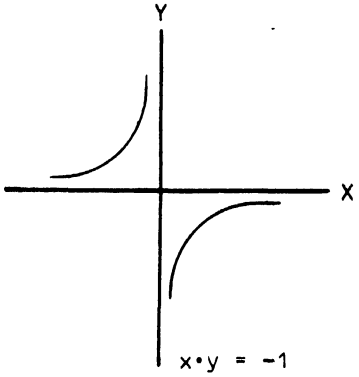




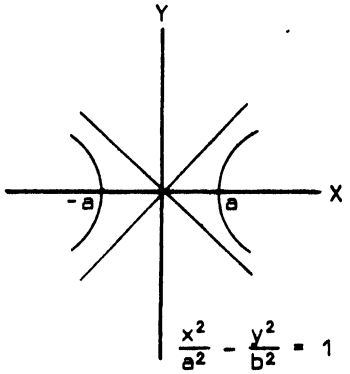




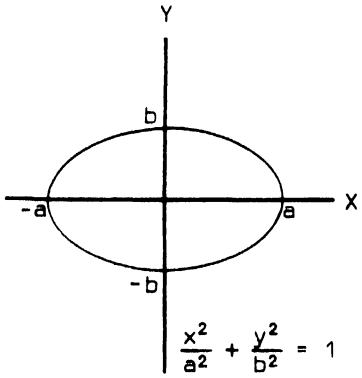
$x \cdot y = 1$



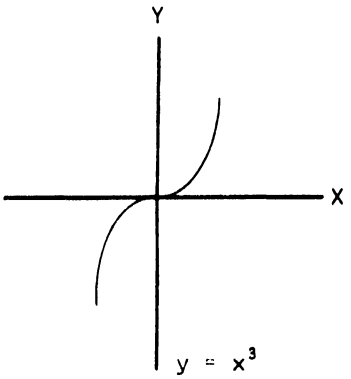
$x \cdot y = -1$



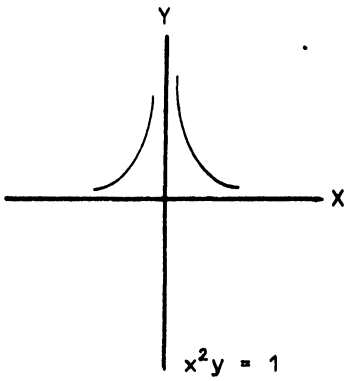
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



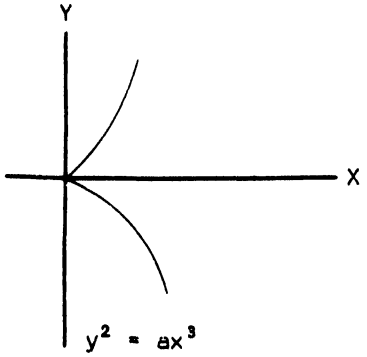
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



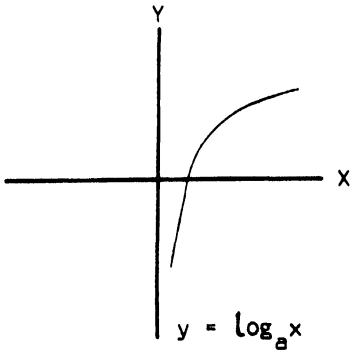
$y = x^3$



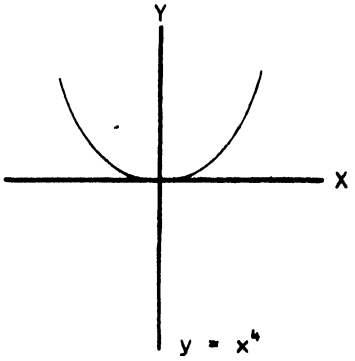
$x^2y = 1$



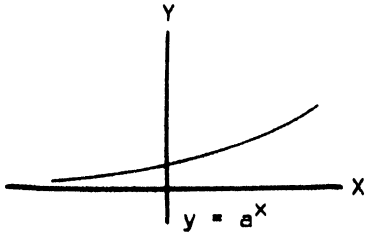
$y^2 = ax^3$



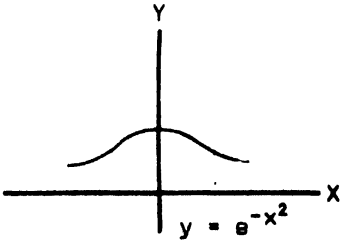
$y = \log_a x$



$y = x^4$



$y = a^x$



$y = e^{-x^2}$

PART II - CALCULUS

1.1 Introduction

Consider the growth of a young fish. In the ordinary way we know that it grows gradually and continuously. If it is examined after an interval of a few months (years), the growth will be obvious and can be readily measured. But if it is observed after an interval of a few days, although growth has taken place, the amount is too small to be distinguished. If observations take place after a still shorter period of time, say a few hours, although no change can be detected, we know that there has been growth, which, using a mathematical expression, can be regarded as infinitesimally small or infinitesimal.

What is of real importance in most cases is not necessarily the actual amount of growth or increase, but the "rate of growth or increase". This is the problem, closely connected as it is with infinitesimal increases, that is the basis of the Calculus (Differential Calculus).

1.2 General notation for functions [1/]

General forms for indicating a function of x are: $F(x)$, $f(x)$, $\phi(x)$.

A statement such as: $f(x) = 2x^2 - 8x + 7$
or, $f(\theta) = \sin^2\theta - \cos^2\theta$
defines the specific functions of the variables concerned.

Thus, if $f(x) = 2x^2 - 8x + 7$, $f(2)$ stands for the numerical value of the functions when 2 is substituted for x:

$$f(2) = 2 \times 2^2 - 2 \times 8 + 7 = -1$$
$$\text{also,} \quad f(1) = 2 \times 1^2 - 2 \times 1 + 7 = 7 , \quad \text{etc.}$$

1.3 Notation for increases in functions

If y is a function of x, $y = f(x)$, and if x is increased by δx, then y will be increased in consequence and its increment will be denoted by δy:

(1) $\quad y = f(x)$

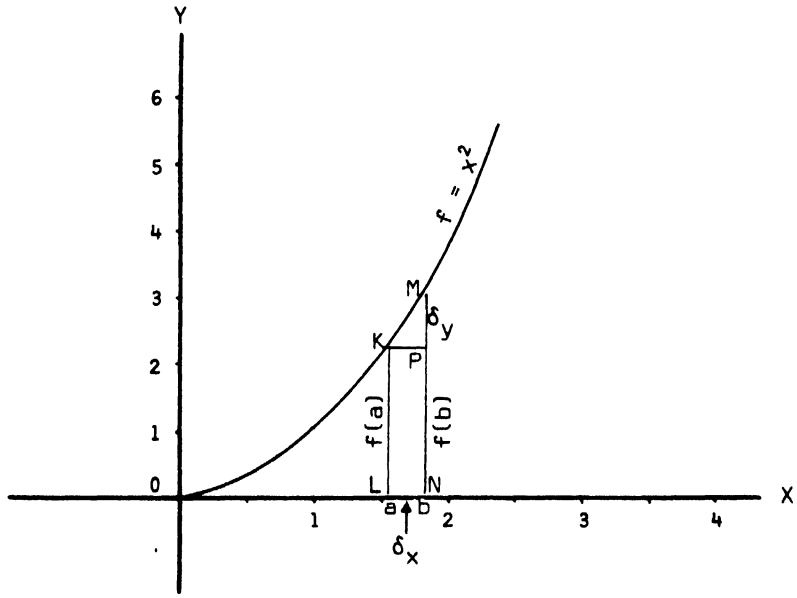
(2) $\quad (y + \delta y) = f(x + \delta x)$

(2)-(1) $\quad \delta y = f(x + \delta x) - f(x)$

1/ See also: Part III - Trigonometry

If $y = x^2 - 14x + 45$ and x receives the increment δx, y will receive the increment δy. Then, $(y + \delta y) = (x + \delta x)^2 - 14(x + \delta x) + 45$

A graphical presentation: Consider the example of $f(x) = x^2$ or $y = x^2$. The figure below presents part of the curve.



As indicated in the above graph, two points L and N are taken on the OX-axis, so that:

$$OL = a \quad , \quad ON = b$$

The corresponding ordinates are:

$$LK = f(a) \quad \text{and} \quad MN = f(b)$$

In general, if L is any point on OX so that $OL = x$, let x be increased by LN where $LN = \delta x$. MP represents the corresponding increase in $f(x)$ or y.

(1) $KL = f(x)$

(2) $MN = f(x + \delta x)$

(2)-(1) $MP = \delta y = f(x + \delta x) - f(x)$

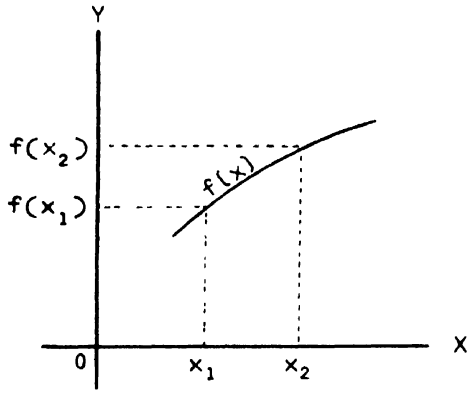
1.4 Variations in functions, limits

A function of x is increasing through a range of numerical values of x if for every pair of values x_1 and x_2, we have:

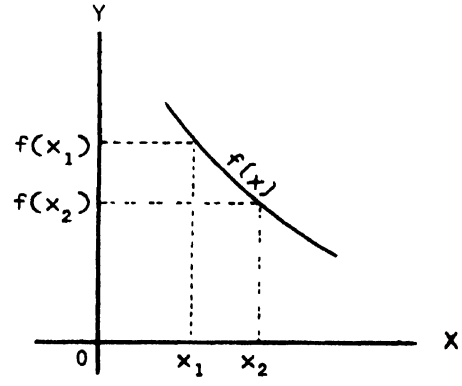
$$(x_2 - x_1)\left(f(x_2) - f(x_1)\right) \geqslant 0$$

A function of x is decreasing through a range of numerical values of x if for every pair of values x_1 and x_2, we have:

$$(x_2 - x_1)\left(f(x_2) - f(x_1)\right) \leqslant 0$$



Increasing function of x

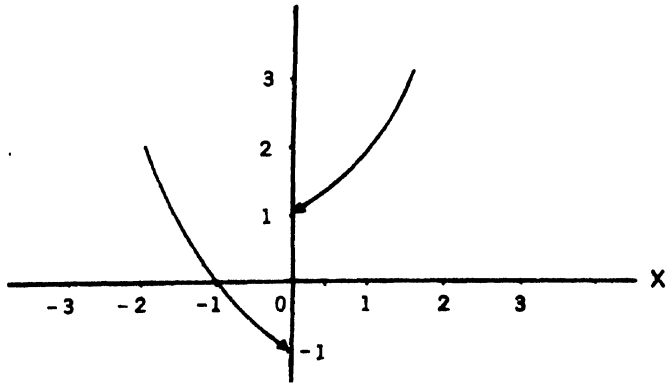


Decreasing function of x

Limits: Here, we examine the behaviour of a function as it approaches a particular point. Consider for example the function:

$$f(x) \begin{cases} x^2 + 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ x^2 - 1 & \text{if} \quad x < 0 \end{cases}$$

The function approaches the value $+1$ as x tends to 0 through positive values, and -1 as x tends to zero through negative values.



These two values, +1 and -1, are sometimes called the "right-hand limit" and the "left-hand limit" of the function as x tends to 0, and we can write:

$$\lim_{x \to 0+} f(x) = +1 \quad , \quad \lim_{x \to 0-} f(x) = -1$$

So the limit (if it exists) of a function as we approach a point is not necessarily the same as the value of the function at the point.

If when x approaches a particular value $x = a$ (whether from the above or below), the values of f(x) approach a certain value ℓ, we call this "the limit of f(x) as x tends to a", and write:

As, $x \to a$, $f(x) \to \ell$

or, $\lim_{x \to a} f(x) = \ell$ or $\lim_{x \to a} f(x) = \ell$

Theorems about limits:

1. For the constant function f, where $f(x) = c$, the $\lim_{x \to a} f(x) = c$

2. If $\lim_{x \to a} f(x) = F$ and $\lim_{x \to a} g(x) = G$, then

$$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = F \pm G$$

3. $$\lim_{x \to a} f(x)\, g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = FG$$

4. $$\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = kF, \quad k \text{ constant}$$

5. $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{F}{G}, \quad \text{if } G \neq 0$$

Exercise 1: Find $\lim_{x \to 2} (3x^3 - 2x^2 + 4)$:

$$\lim_{x \to 2} (3x^3 - 2x^2 + 4) = \lim_{x \to 2} (3x^3) - \lim_{x \to 2} (2x^2) + \lim_{x \to 2} (4)$$

$$= 3 \times 8 - 2 \times 4 + 4 = 20$$

Exercise 2: Find $\lim_{x \to 1} \frac{x^2 + 3}{x + 2} = \frac{\lim_{x \to 1} (x^2 + 3)}{\lim_{x \to 1} (x + 2)} = \frac{4}{3}$

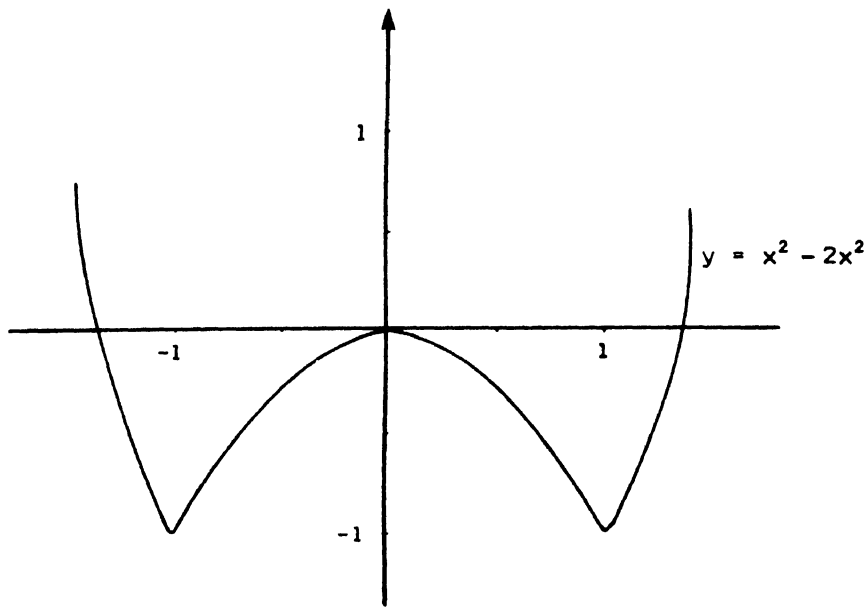
1.5 Continuity

The concept of continuity is extremely important in mathematics and needs careful definition. For the present we shall simply say that a function is continuous at a particular point if we can depart the point along the graph both to the left and to the right without having to make a jump.

1.6 Graphs and Continuity

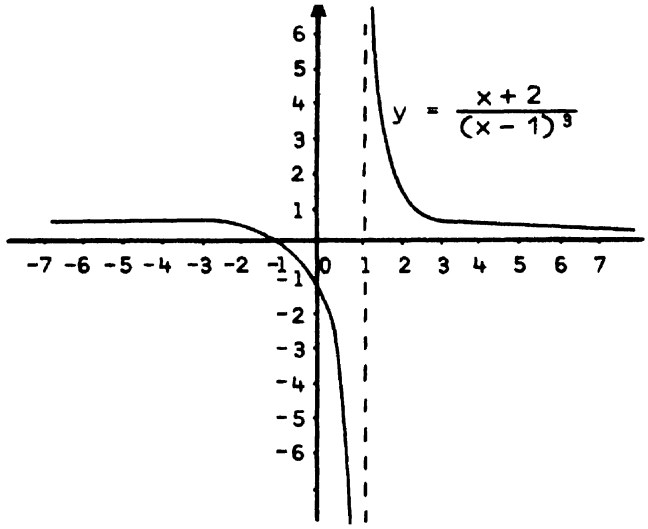
1. Polynomial Functions:

A polynomial function, defined by $y = P(x)$, where $P(x)$ is a polynomial is continuous everywhere. The graph of a polynomial function is a continuous curve. The domain of the definition is the set of real numbers; the range is a sub-set of the real numbers (which could be the whole set).



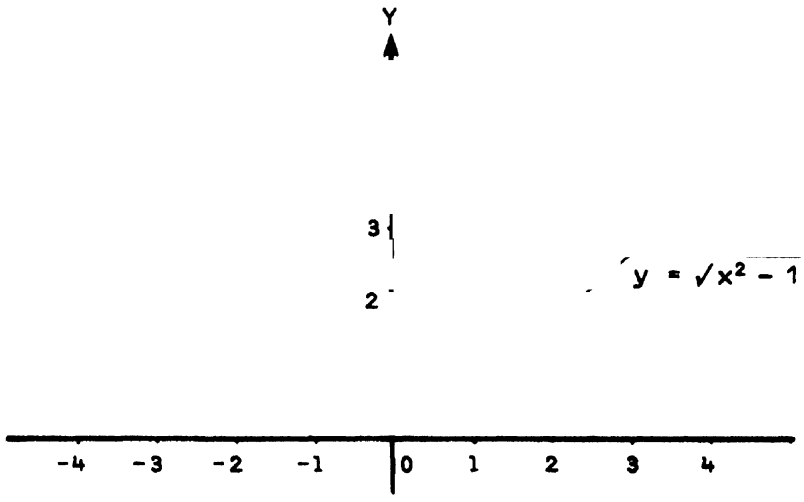
Rational functions:

A rational function, defined by $y = \frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials, and Q(x) is not the zero polynomial, is continuous everywhere with the exception of, at most, a finite number of isolated values of x, namely those for which Q(x) = 0. These values must be excluded from the domain. The range is a sub-set of the reals.



3. Explicit Algebraic Functions:

Explicit algebraic functions, as defined, include the rational functions as special cases. An explicit algebraic function may, therefore, fail to be continuous. In addition it may have isolated points. The domain and the range are sub-sets of the reals.



1.7 Differential Coefficient, Differentiation

The algebraical aspect of the rate of change of a function:

1. The value of a function changes as the variable changes upon which it depends.

2. The rate of change (whether of increase or decrease) can be found geometrically. Specifically:

   a) When the function is of the first degree, it is represented by a straight line and the gradient of this straight line is equal to the rate of change of the function. If $y = f(x)$ (linear), and δx and δy are corresponding increases of x and y, the gradient is equal to $\frac{\delta y}{\delta x}$. This is constant throughout the line, i.e. the rate of change is uniform.

   b) When the function is not of the first degree, its graph will be a curve, and the rate of change of the function will differ in different parts of the curve. Its value at any point is equal to the gradient of the tangent at the point on the curve corresponding to any assigned value of x.

The geometrical method has many important applications. However, for practical purposes and for accuracy, an algebraic method is necessary:

1. If y is a continuous function of x, and δx is any increase in the value of x, there will be a corresponding increase (or decrease) in the value of y, denoted by δy.

2. The ratio $\frac{\delta y}{\delta x}$ represents the average rate of increase.

3. Since y is a continuous function of x, if δx becomes infinitely small, so also does δy.

4. When $\delta x \to 0$, the ratio $\frac{\delta y}{\delta x}$ in general tends to a finite limit, and this limit is called the differential coefficient of y (function) with respect to x (independent variable). It is represented by the symbol:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

The differential coefficient $\frac{dy}{dx}$ is also called a derivative of y with respect to x, or the derived function. It should be noted that, except in the case of a linear function, the differential coefficient of y with respect to x is itself a function of x. Furthermore, the process of finding the differential coefficient or derivative of a function is called Differentiation. In general, the differentiation of f(x) with respect to x can be expressed by:

$$\frac{\left(f(x)\right)}{dx} \qquad \text{or} \qquad \frac{d}{dx} f(x) \qquad \text{or} \qquad D_x y$$

Example:

Let (1) $y = x^2$

Then (2) $y + \delta y = (x + \delta x)^2$

(2)-(1) $\delta y = (x + \delta x)^2 - x^2$

$= 2x(\delta x) + (\delta x)^2$

Dividing by δx, $\frac{\delta y}{\delta x} = 2x + \delta x$

When $\delta x \rightarrow 0$, $\frac{\delta y}{\delta x}$ approaches a limit

and, $\underset{\delta x \rightarrow 0}{\text{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx} = 2x$

The $\frac{dy}{dx}$ represents the rate of increase of y with respect to x, for any assigned value of x. For example, when $x = 1$, $\frac{dy}{dx} = 2$, i.e. the rate of increase of y, or x^2 with respect to x is 2.

General definition of the differential coefficient:

It has been seen that the differential coefficient of a function is equal to the gradient of the tangent at a point on a curve which represents the function. The general expression for the differential coefficient of any function, f(x) is given by:

$$\underset{\delta x \rightarrow 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x}$$

It should be noted that, as the gradient may be positive or negative, the calculated differential coefficient may be positive or negative.

It should also be noted that, derivative, absolute instantaneous rate and slope of the tangent are equivalent concepts expressed from different view points.

Remarks:

1. The derivative is not the value of $\delta y/\delta x$, when $\delta x = 0$; for when $\delta x = 0$, $\delta y/\delta x = {}^{0}/_{0}$, which is indeterminate.

2. The derivative is the limit of $\delta y/\delta x$ as $\delta x \to 0$ if this limit exists. However, the limit may fail to exist, and in this case the derivative is not defined. For polynomials, however, the derivative exists for every value of x.

Graphical presentation of derivatives:

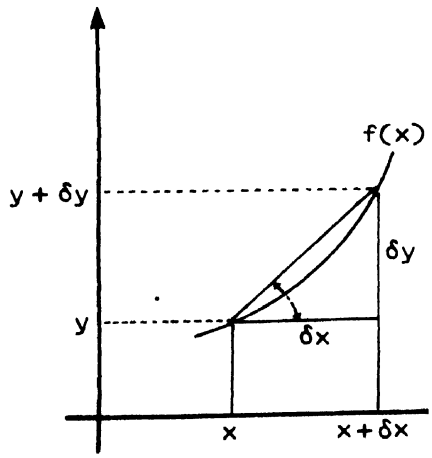


Figure 1

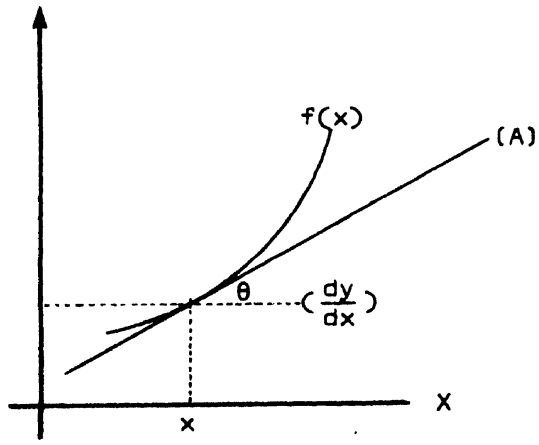


Figure 2

In Figure 1, the increment ratio $\frac{\delta y}{\delta x}$ is the slope of the secant to the curve passing through the points (x,y) and $(x + \delta x, y + \delta y)$.

In Figure 2, $\frac{dy}{dx}$ is the slope of the tangent (A) to the curve at the point (x,y), the tangent is the limit of the secant when δx tends to zero and $x + \delta x$ tends to x.

1.8 General Differentiation

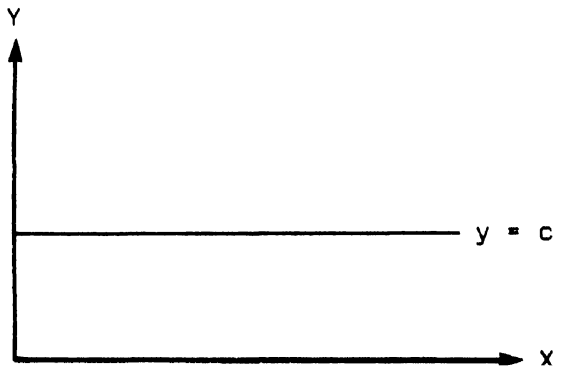
1. The differential coefficient of a constant:

Let $y = c$, constant. Then any change in x, say δx, will not affect y, i.e. $dy = 0$.

$$\therefore \quad \frac{\delta y}{\delta x} = 0, \quad \text{i.e.} \quad \frac{dy}{dx} = 0$$

The differential coefficient of a constant is zero,

$$\frac{d}{dx}(c) = 0 \quad \text{or} \quad \frac{dc}{dx} = 0$$



The geometrical presentation of the function is given in the graph above. The equation, $y = c$, represents a straight line parallel to the X-axis. The gradient of the line is zero, that is, $\frac{dy}{dx} = 0$.

2. Differentiation of $y = ax + b$:

By applying the steps of differentiation, we get:

(1) $\quad y = ax + b$

(2) $\quad y + \delta y = a(x + \delta x) + b$

(2)-(1) $\quad y = a(\delta x)$

$$\frac{\delta y}{\delta x} = a$$

Since a is constant, $\frac{dy}{dx} = a$, i.e. $\frac{d}{dx}(ax + b) = a$.

The graph of $y = ax + b$ is a straight line and therefore of a constant gradient.

3. Differentiation of $y = x^3$

(1) $y = x^3$

(2) $y + \delta y = (x + \delta x)^3 = (x^3 + 3x^2\delta x + 3x\delta^2 x + \delta^3 x)$

(2)-(1) $\delta y = 3x^2\delta x + 3x\delta^2 x + \delta^3 x$

$$\frac{\delta y}{\delta x} = 3x^2 + 3x\delta x + \delta^2 x$$

$$\frac{dy}{dx} = 3x^2 \quad \text{i.e.} \quad \frac{d}{dx}(x^3) = 3x^2$$

4. Differentiation of $y = x^n$ where n is positive:

We lead to the general case of $\frac{d}{dx}(x^n)$ by means of the special cases:

(i) $\frac{d}{dx}(x) = 1$

(ii) $\frac{d}{dx}(x^2) = 2x$

(iii) $\frac{d}{dx}(x^3) = 3x^2$

(iv) $\frac{d}{dx}(x^4) = 4x^3$

(v) $\frac{d}{dx}(x^5) = 5x^4$

The above results suggest the rule for differentiating x^n,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

5. Differentiation of x^n when n is a negative number:

Example 1:

Let $y = x^{-1} = \frac{1}{x}$

Applying the steps of differentiation, we get:

(1) $y = \frac{1}{x}$

(2) $y + \delta y = \frac{1}{x + \delta x}$

(2)-(1) $\delta y = \frac{1}{x + \delta x} - \frac{1}{x} = \frac{-\delta x}{x(x + \delta x)}$

$$\frac{\delta y}{\delta x} = -\frac{1}{x(x + \delta x)}$$

Let $\delta x \longrightarrow 0$

$$\frac{dy}{dx} = -\frac{1}{x^2} = (-1)x^{-2}$$

Example 2:

Let $y = x^{-2} = \frac{1}{x^2}$

(1) $y = \frac{1}{x^2}$

(2) $y + \delta y = \frac{1}{(x + \delta x)^2}$

(2)-(1) $\delta y = \frac{1}{(x + \delta x)^2} - \frac{1}{x^2} = \frac{-2x\delta x - (\delta x)^2}{x^2(x + \delta x)^2}$

$$\frac{\delta y}{\delta x} = \frac{-2x - \delta x}{x^2(x + \delta x)^2}$$

Let $\delta x \longrightarrow 0$

$$\frac{dy}{dx} = -\frac{2x}{x^4} = -\frac{2}{x^3} \quad \text{i.e.} \quad \frac{d}{dx}(x^{-2}) = -2x^{-3}$$

Similarly, $\frac{d}{dx}(x^{-3}) = -3x^{-4}$

These results suggest the rule for differentiating x^{-n},

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

Example:

Differentiate with respect to x, $y = \frac{1}{x^5}$

$$\frac{d}{dx}\left(\frac{1}{x^5}\right) = \frac{d}{dx}(x^{-5}) = -5x^{-5-1} = -5x^{-6} = -\frac{5}{x^6}$$

6. Differentiation of cx^n:

$$y = cx^n$$

$$\frac{d}{dx}(cx^n) = c\left(\frac{d}{dx}x^n\right) = cnx^{n-1}$$

Example 1:

If $y = 4x^2$, differentiate with respect x.

$\therefore$ $$y = 4x^2$$

$$\frac{d}{dx}(4x^2) = 4\left(\frac{d}{dx}(x^2)\right) = 4 \times (2x) = 8x$$

Example 2:

Differentiate with respect x, $y = \frac{5}{x^3}$

$$\frac{5}{x^3} = 5(x^{-3})$$

$$\frac{d}{dx}\left(5(x^{-3})\right) = 5\left(\frac{d}{dx}(x^{-3})\right)$$

$$5(-3x^{-3-1}) = -15x^{-4} = -\frac{15}{x^4}$$

Example 3:

Differentiate with respect x, $y = -\frac{5}{x^6}$

$$y = \frac{-5}{x^6} = -5(x^{-6})$$

$$\frac{d}{dx}\left(-5(x^{-6})\right) = -5\frac{d}{dx}(x^{-6})$$

$$= -5(-6x^{-6-1}) = 30x^{-7} = \frac{30}{x^{-7}}$$

1.9 Some rules for Differentiation

1.9.1 Differentiation of sums

Let u and v be functions of x

Let y be their sum, so that $y = u + v$

The differential coefficient of the sum for a number of functions is equal to the sum of the differential coefficients of these functions,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Example 1:

Differentiate with respect to x,

$$y = 4x^3 + 7x^2 - 10x + 5$$

Using the above rule,

$$\frac{dy}{dx} = 12x^2 + 14x - 10$$

Example 2:

Find the gradient at that point on the curve $y = x^2 - 5x + 4$, where $x = 4$

$\therefore$ $$y = x^2 - 5x + 4$$

$$\frac{dy}{dx} = 2x - 5$$

When $x = 2$,

$$\frac{dy}{dx} = 2 \times 4 - 5 = 3$$

Note that, when the gradient is zero,

$$2x - 5 = 0$$

$$x = 5/2$$

Example 3:

Find $\frac{dy}{dx}$ when $y = \sqrt{x} + \sqrt[3]{x} + \frac{2}{x}$

$$= x^{1/2} + x^{1/3} + 2x^{-1}$$

$$\frac{dy}{dx} = 1/2\, x^{1/2-1} + 1/3\, x^{1/3-1} + 2\,(-1x^{-1-1})$$

$$1/2\, x^{-1/2} + 1/3\, x^{-2/3} - 2x^{-2}$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}} \qquad x^2$$

Example 4:

Find the gradient at that point on the curve of $y = 2x^2 - 3x + 1$, where $x = 1.5$. For what value of x will the curve have a zero gradient.

$$y = 2x^2 - 3x + 1$$

(i) $$\frac{dy}{dx} = 4x - 3$$

When, $$x = 1.5,$$

$$\frac{dy}{dx} = 4 \times 1.5 - 3 = 3$$

(ii) When the gradient is zero

$$4x - 3 = 0$$

$$x = 3/4$$

1.9.2 Differentiation of a product

A general rule for use in all cases is as follows:

Let u and v be functions of x.

Let $y = u \times v$.

Consequently, y is also a function of x.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Rule:

(i) Differentiate each factor in turn and multiply by the other factor

(ii) The sum of the products is $\frac{dy}{dx}$

Example 1:

Find the differential coefficient of the product $y = uvw$ where u,v,w are functions of x.

$$\frac{dy}{dx} = \frac{du}{dx}vw + \frac{dv}{dx}uw + \frac{dw}{dx}uv$$

Example 2:

Differentiate the product,

$$\therefore \quad y = (x^2 + 1)(x + 1)$$

$$\frac{dy}{dx} = \left(\frac{d}{dx}(x^2 + 1)\right)(x + 1) + \left(\frac{d}{dx}(x + 1)\right)(x^2 + 1)$$

$$= 2x(x + 1) + (x^2 + 1)$$

$$= 2x^2 + 2x + x^2 + 1 = 3x^2 + 2x + 1$$

1.9.3 Differentiation of a quotient

A general rule for use in all cases is as follows:

Let u and v be functions of x.

Let $y = \frac{u}{v}$

Consequently, y is a function of x.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The above expression can be written,

$$\frac{dy}{dx} = \frac{(\text{den.} \times \text{d.c.of num.}) - (\text{num.} \times \text{d.c.of den.})}{(\text{den.})^2}$$

Example:

Differentiate $\frac{(x^2 + x)}{x^2 - 3x + 2}$

$$y = \frac{(x^2 + x)}{x^2 - 3x +}$$

$$\frac{dy}{dx} = \frac{(x^2 - 3x + 2)(2x + 1) - (x^2 + x)(2x - 3)}{(x^2 - 3x + 2)^2}$$

$$\frac{-4x^2 + 4x + 2}{(x^2 - 3x + 2)^2}$$

2.0 The function of a function

Suppose we are given y as a function of the variable u, say $3u^5$, and u as some function of x, say $u = 1 - x^2$, and we require to obtain $\frac{dy}{dx}$

The general rule of differentiating the function of a function is:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Let $y = 3u^5$ and $u = 1 - x^2$.

By applying the above rule, we have:

$$\frac{dy}{du} = 15u^4 \quad , \quad \frac{du}{dx} = -2x$$

and,

$$\frac{dy}{dx} = 15u^4 \times (-2x)$$

$$= 15(1 - x^2)^4 \times (-2x)$$

$$= -30x(1 - x^2)^4$$

Example 1:

Differentiate $(1 - 4x^2)^8$

$$y = (1 - 4x^2)^8$$

$$u = 1 - 4x^2$$

$$y = u^8$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 8u^7 \times (-8x)$$

$$= -64x(u)^7 = -64x(1 - 4x^2)^7$$

Example 2:

Differentiate $\sqrt{4 - x^2}$

$$y = (4 - x^2)^{\frac{1}{2}}$$

$$u = 4 - x^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{2} u^{-\frac{1}{2}} \times (-2x)$$

$$= \frac{1}{2} \frac{1}{\sqrt{u}} \times (-2x)$$

$$\frac{1}{2\sqrt{4-x^2}} \times (-2x) = \frac{-x}{\sqrt{4-x^2}}$$

Example 3:

Differentiate $y = (3x^2 - 5x + 4)^{3/2}$

$$u = 3x^2 - 5x + 4$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{3}{2} u^{\frac{1}{2}} \times (6x - 5)$$

$$\frac{3}{2} \left(\sqrt{3x^2 - 5x + 4}\right) \times (6x - 5)$$

e 4:

Differentiate $\frac{\sqrt{1 + 3x}}{4x}$

4x

$$\frac{du}{dx} = \frac{d}{dx} (1 + 3x)^{\frac{1}{2}} = \frac{1}{2} (1 + 3x)^{-\frac{1}{2}} \times 3 = \frac{3}{2\sqrt{1 + 3x}}$$

$$\frac{dv}{dx} = \frac{d}{dx} (4x) = 4$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{1}{16x^2}\left(4x \times \frac{3}{2\sqrt{1+3x}} - 4\sqrt{1+3x}\right)$$

$$\frac{6x - 4(1+3x)}{16x^2\sqrt{1+3x}} = \frac{-4-6x}{16x^2\sqrt{1+3x}}$$

$$\frac{2+3x}{8x^2\sqrt{1+3x}}$$

2.1 Differentiation of implicit functions

In the case of implicit functions, two variables x and y occur in the form of an equation from which y can be obtained in terms of x, though sometimes this is not possible.

Differentiate the following equation:

$$3x^2 - 2y^2 + x = 4y$$

Differentiating, we get:

$$6x - 4y\frac{dy}{dx} + 1 = 4\frac{dy}{dx}$$

The above equation can be solved for $\frac{dy}{dx}$:

$$4y\frac{dy}{dx} + 4\frac{dy}{dx} = 6x + 1$$

$$\frac{dy}{dx}(4y + 4) = 6x + 1$$

$$\frac{dy}{dx} = \frac{6x+1}{4y+4}$$

When numerical values of x and y are known, the numerical value of $\frac{dy}{dx}$ can de determined.

Example:

Find the slope of the tangent to the curve $2x^2 + xy + y^2 = 4$ at point $(2, -2)$.

$\therefore$ Differentiating the above equation (remember xy is a product), we get:

$$4x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x + 2y) = -(4x + y)$$

$$\frac{dy}{dx} = -\frac{4x + y}{x + 2y}$$

When, $x = 2$, $y = -2$

$$\frac{dy}{dx} = -\frac{(4 \times 2) - 2}{2 - 4}$$

$$-\frac{6}{-2} = 3$$

The gradient of the tangent to the curve at this point is 3 and the angle of the slope is $71°34'$.

2.2 Higher derivations

Suppose y is a given function of x,

$$y = 3x^5 - x^3$$

Differentiating, we get:

$$\frac{dy}{dx} = 15x^4 - 3x^2$$

We denote, $\frac{dy}{dx} = y'$

so, $y' = 15x^4 - 3x^2$

We call y' the first differential coefficient of y with respect to x.

y' is a definite function of x, and in the ordinary way, when differentiating y' with respect to x, we get:

$$\frac{dy'}{dx} = 60x^3 - 6x$$

We denote, $$\frac{dy'}{dx} = y'' = \frac{d^2y}{dx^2}$$

and call it the second differential coefficient of y with respect to x.

If we differentiate $\frac{d^2y}{dx^2}$ with respect to x, we get the third differential coefficient denoted by:

$$\frac{d^3y}{dx^3} = 180x^2 - 6 \quad , \quad \text{etc.}$$

∴ Successive differential coefficients of x^n :

$$y = x^n$$

$$\frac{dy}{dx} = nx^{(n-1)}$$

$$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3}$$

Note: $n! = n(n-1)(n-2) \ldots 2 \times 1$

2.2.1 Derived curves

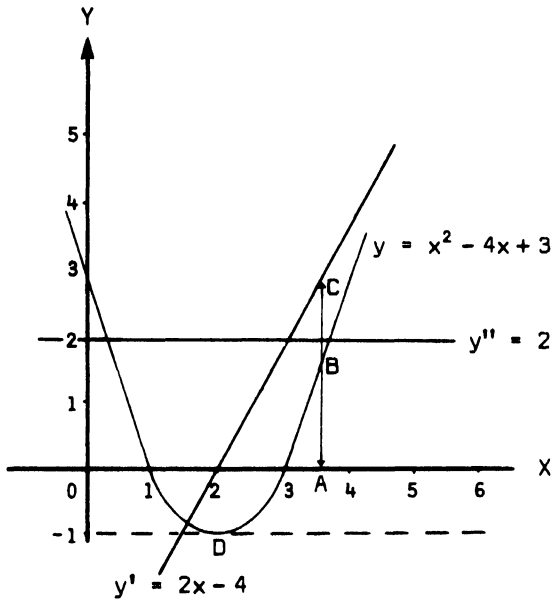
It has been seen that successive differentiation of a function of x produces a set of derivatives, each of which is also a function of x. A graphical presentation of the curves of the derived functions and of the existing relations between them is given below.

Consider the function, $$y = x^2 - 4x + 3$$

Then, $$y' = \frac{dy}{dx} = 2x - 4$$

$$\frac{d^2y}{dx^2}$$

The figure below gives the curves of the above three functions.



$y = x^2 - 4x + 3$

x	-1	0	1	2	3	4
y	8	3	0	-1	0	3

$y' = 2x - 4$

x	1	2	3
y	-2	0	2

$\therefore$ Since y', the first derived function, gives the rate of increase of y with respect to x, its value for any assigned value of x equals the gradient of the corresponding point of the curve.

Take for example point A on the OX, where $x = 3.5$. Drawing the ordinate at A, B is the corresponding point on the curve, and C the point on the straight line with function $y' = 2x - 4$.

The value of the ordinate AC is equal to the gradient of the curve ($y = x^2 - 4x + 3$) at B. The corresponding values of AC and point B are:

$$AC = (2 \times 3.5 - 4) = 3$$

$$B_{(x,y)} = B\ (3.5,\ 1.25)$$

$\therefore$ From the graph it is indicated that $y'' = 2$, being parallel to OX and having a constant value for any ordinate, shows that the gradient of $y' = 2x - 4$ is constant, viz: 2.

$\therefore$ The curve ($y = x^2 - 4x + 3$) has its lowest point D at $x = 2$. The value of the corresponding point of $y' = 2x - 4$ is $y' = 2 \times 2 - 4 = 0$. This in turn indicates that the gradient of the original function is zero, when $x = 2$. A tangent drawn to the curve at D will be parallel to OX.

$\therefore$ Note that, for values of x less than 2, the function $y = x^2 - 4x + 3$ is decreasing, while the derived function values are negative. For values greater than 2, $x^2 - 4x + 3$ is increasing and the derived function $2x - 4$ is positive.

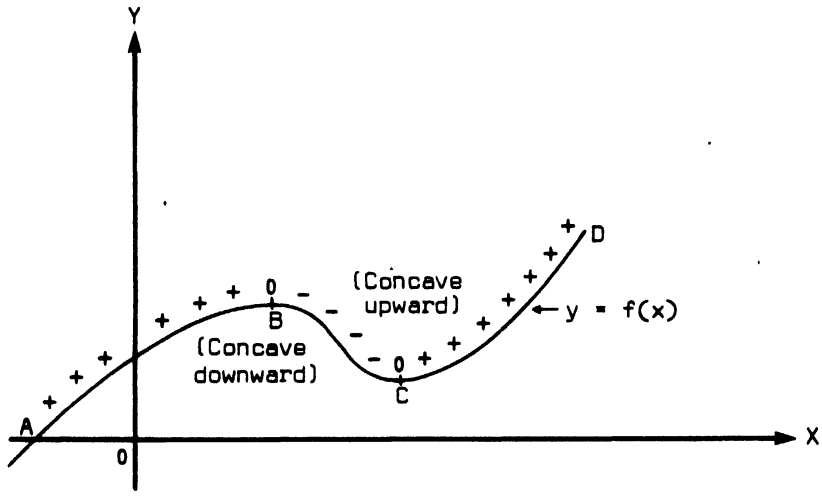
2.2.2 Sign of the derivative

When y is given as a function of x, we have seen that $\frac{dy}{dx}$ gives the rate of increase of y with respect to x.

Hence, if $\frac{dy}{dx}$ is positive, y increases as x increases, and if $\frac{dy}{dx}$ is negative, y decreases as x increases.

In the same way, as $\frac{d^2y}{dx^2}$ is the differential coefficient of $\frac{dy}{dx}$ with respect to x, if $\frac{d^2y}{d^2x}$ is positive, $\frac{dy}{dx}$ increases as x increases, and if $\frac{d^2y}{dx^2}$ is negative, $\frac{dy}{dx}$ decreases as x increases.

2.3 Maxima and Minima



Consider the curve AD portrayed in the above figure.

For the portion AB of the curve, y increases as x increases and so $\frac{dy}{dx}$ is positive. At point B, $\frac{dy}{dx}$ is zero.

Along BC, $\frac{dy}{dx}$ is negative and is again zero at C.

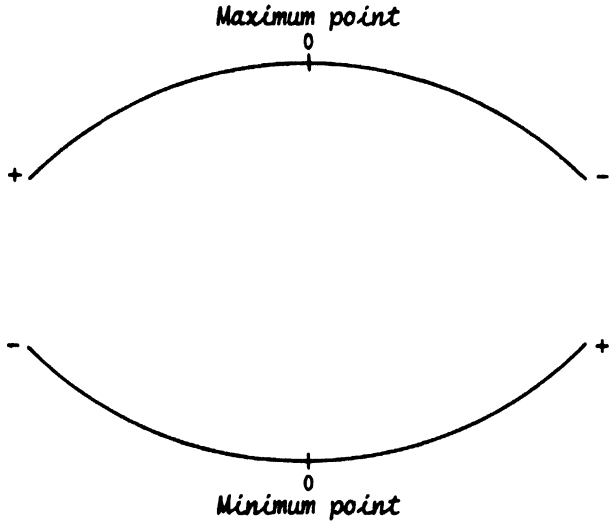
For the portion CD, $\frac{dy}{dx}$ is again positive.

The points B and C of the curve are called turning points.

The ordinate at B is greater than any neighbouring ordinate, and B is called a maximum point of the curve.

The ordinate at C is less than any neighbouring ordinate, and C is called a minimum point of the curve.

2.3.1 Method for determining maximum and minimum points



Maximum point: (1) $\frac{dy}{dx} = 0$

(2) $\frac{dy}{dx}$ changes from positive to negative in passing through the point, i.e.
$\frac{dy}{dx}$ is decreasing.
$\therefore \frac{d^2y}{dx^2}$ is negative.

Minimum point: (1) $\frac{dy}{dx} = 0$

(2) $\frac{dy}{dx}$ changes from negative to positive in passing through the point, i.e.
$\frac{dy}{dx}$ is increasing.
$\therefore \frac{d^2y}{dx^2}$ is positive.

Example 1:

Determine the turning point of the curve $y = 3x^2 - x + 1$; find whether it is a maximum or a minimum point.

(1) $$\frac{dy}{dx} = 6x - 1$$

But for the turning point,

$$\frac{dy}{dx} = 0 \quad , \quad 6x - 1 = 0$$
$$x = \tfrac{1}{6}$$

So when $x = \frac{1}{6}$ there is a possible turning point.

(2) To find whether this is a maximum or a minimum point, we consider the sign of $\frac{d^2y}{dx^2}$ at this point.

We have, $\frac{d^2y}{dx^2} = 6$

$\therefore$ When, $x = \frac{1}{6}$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 6$ (positive)

$\therefore$ The point at which $x = \frac{1}{6}$ is a minimum point of the curve.

$\therefore$ Minimum value of $y = 3(\frac{1}{6})^2 - \frac{1}{6} + 1 = \frac{11}{12}$

Example 2:

Find the turning points of the curve $y = 4x^3 - 3x^2 - 18x + 6$ and distinguish between them:

(1) $$\frac{dy}{dx} = 12x^2 - 6x - 18$$

For the turning point,

$$12x^2 - 6x - 18 = 0$$
$$(2x - 3)(x + 1) = 0$$
$$x_1 = \tfrac{3}{2} \quad , \quad x_2 = -1$$

(2) $$\frac{d^2y}{dx^2} = 24x - 6$$

$\therefore$ When, $x = \frac{3}{2}$, $\frac{d^2y}{dx^2} = 30$ (positive)

The point at which $x = \frac{3}{2}$ is a minimum point.

$\therefore$ When, $x = -1$, $\frac{d^2y}{dx^2} = -30$ (negative)
The point at which $x = -1$ is a maximum point.

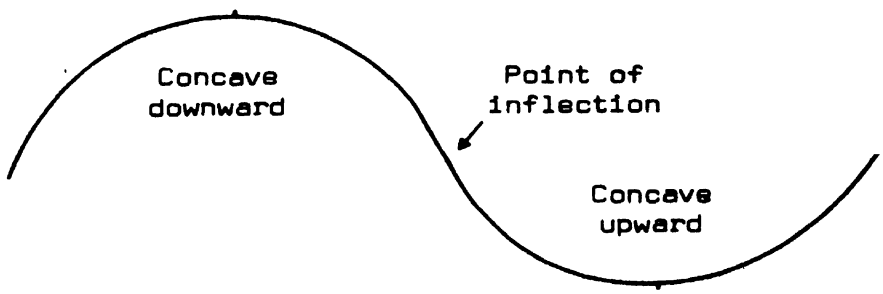
$\therefore$ The point $P_1\ (3/2, -\frac{57}{4})$ is a minimum point.
The point $P_2\ (-1, 17)$ is a maximum point.

2.3.2 Concavity and convexity : Points of inflection

We have seen that the sign $\frac{d^2y}{dx^2}$ determines whether $\frac{dy}{dx}$ is increasing or decreasing, i.e. whether the curve $y = f(x)$ is concave upwards or downwards (i.e. convex downwards or upwards).

Alternatively, we can think of x increasing in the first stretch as a bend taken with a "right-hand lock", followed in the second stretch, as a bend taken with a "left-hand lock". The point where the lock changes (and the curve crosses its tangent, and $\frac{d^2y}{dx^2}$ changes sign) is called a point of inflection.

$\frac{d^2y}{dx^2} < 0 \rightarrow \frac{dy}{dx}$ is decreasing



$\frac{d^2y}{dx^2} > 0 \rightarrow \frac{dy}{dx}$ is increasing

Example:

For what values of x is the function $y = x^3 - 3x^2 + 4$:

1. Increasing, decreasing, stationary

2. Concave upwards, concave downwards, at a point of inflection

$\therefore \quad y = x^3 - 3x^2 + 4$

$\frac{dy}{dx} = 3x^2 - 6x = 3x(x - 2)$

And the sign of $\frac{dy}{dx}$ is given by:

x:	-3	-2	-1	0	1	2	3	4	5
$\frac{dy}{dx}$:	+	+	+	+	+	+	0	-	-	-	-	0	+	+	+	+	+	+

Hence, $y = x^3 - 3x^2 + 4$ is increasing if $x < 0$ or $x > 2$
is decreasing if $0 < x < 2$
is stationary if $x = 0$ or $x = 2$

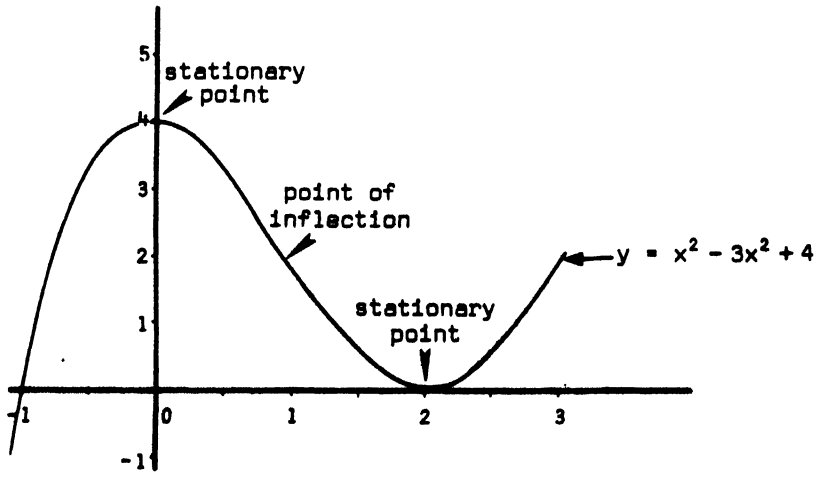
Also, $\frac{d^2y}{dx^2} = 6x - 6 = 6(x - 1)$

And the sign of $\frac{d^2y}{dx^2}$ is given by:

x:	-3	-2	-1	0	1	2	3	4	.5
$\frac{d^2y}{dx^2}$:	-	-	-	-	-	-	-	-	-	0	+	+	+	+	+	+	+	+	+

Hence, $y = x^3 - 3x^2 + 4$ is concave upwards when $x > 1$
is concave downwards when $x < 1$
at a point of inflection when $x = 1$

These can all be illustrated in the graph below of $y = x^3 - 3x^2 + 4$



2.4 Exponential and Logarithmic functions (differentiation)

$\therefore$ $y = e^x$ is called an exponential function because the index or exponent is the variable part of the function.

The value of the constant $e = 2.71828$,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$\therefore$ The function e^x can be expressed in a series involving ascending powers of x:

$$e^x = \left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}$$

$$= 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We also have:

$$e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^{ax} = 1 + \frac{ax}{1} + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \cdots$$

$$e^{-ax} = 1 - \frac{ax}{1} + \frac{a^2x^2}{2!} - \frac{a^3x^3}{3!} + \cdots$$

2.4.1 Differentiation of e^x

Since,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{d}{dx}(e^x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

But this is a series for e^x.

If $y = e^x$, $\frac{d}{dx}(e^x) = e^x$

and that the differential coefficient of e^x is equal to itself.

Similarly, we have:

$$\frac{d}{dx}(e^{-x}) = -e^{-x}$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(e^{-ax}) = -ae^{-ax}$$

2.4.2 The exponential curve

(1) If, $y = e^x$

$$\frac{dy}{dx} = e^x$$

$$\frac{d^2y}{dx^2} = e^x$$

Since $\frac{dy}{dx}$ is always positive, the curve of the function e^x must be positive and always increasing; it has no turning points.

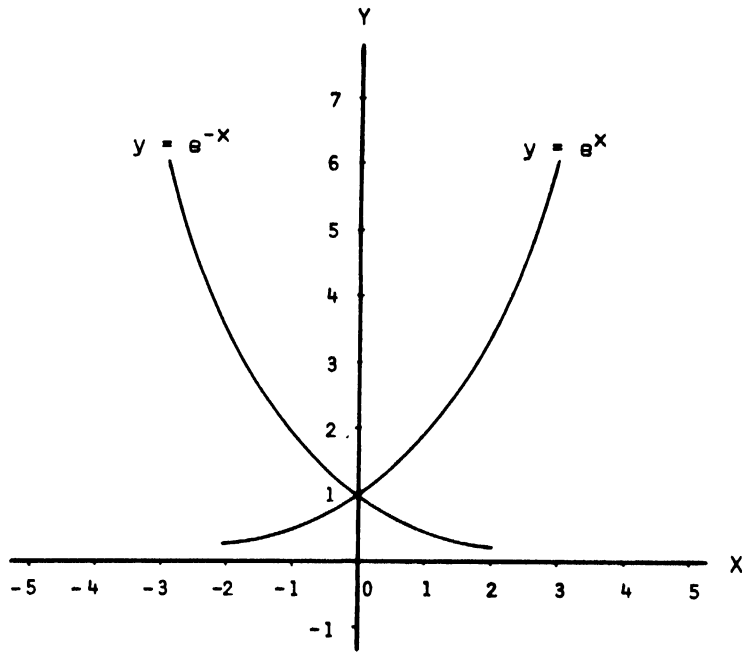
Since $\frac{d^2y}{dx^2} = e^x$, this does not disappear for any finite value of x; there is no point of inflection.

(2) $y = e^{-x}$

$$\frac{dy}{dx} = -e^{-x}$$

$$\frac{d^2y}{dx^2} = e^{-x}$$

$\therefore$ Since $\frac{dy}{dx}$ is always negative, the curve of the function e^{-x} is always decreasing. There are no turning points and no point of inflection.



2.4.3 Differentiation of $\log_e x$

$\therefore$ The differential coefficient of $\log_e x$ (lnx) can be obtained as follows:

Let, $y = \log_e x$

Then, $x = e^y$

$\frac{dx}{dy} = e^y$

And, $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$, $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

$\therefore$ If the logarithm involves a different base, we have:

$$\log_a x = \log_e x \times \log_a e$$

Thus, if $y = \log_a x$

$$= \log_e x \times \log_a e$$

and $$\frac{dy}{dx} = (\log_e x)' \log_a e + (\log_a e)' \log_e x$$

$$= \frac{1}{x} \log_a e + 0 \times \log_e x$$

or $$\frac{dy}{dx} = \frac{1}{x} \log_a e$$

As a special case, if $y = \log_{10} x$

$$\frac{dy}{dx} = \frac{1}{x} \log_{10} e$$

$$= \frac{0.4343}{x}$$

2.4.4 Differentiation of the general exponential function

Let $y = a^x$

Then $\log_e y = x \log_e a$

or $$x = \log_e y \times \frac{1}{\log_e a}$$

$$\frac{dx}{dy} = \frac{1}{y} \times \frac{1}{\log_e a}$$

and $$\frac{dy}{dx} = y \times \log_e a$$

or $$\frac{d}{dx}(a^x) = a^x \log_e a$$

As a special case, $y = 10^x$

$$\frac{d}{dx}(10^x) = 10^x \log_e 10$$

$$= 2.3026 \times 10^x$$

$\therefore$

$$y = e^x \quad , \quad \frac{d}{dx}(e^x) = e^x$$

$$y = e^{-x} \quad , \quad \frac{d}{dx}(e^{-x}) = -e^{-x}$$

$$y = a^x \quad , \quad \frac{d}{dx}(a^x) = a^x \log_e a$$

$$y = \log_e x \quad , \quad \frac{d}{dx}(\log_e x) = \frac{1}{x}$$

Example 1:

Differentiate $y = e^{2x^3}$:

$\therefore$

$$y = e^{2x^3}$$

$$\frac{dy}{dx} = e^{2x^3} \times \frac{d}{dx}(2x^3)$$

$$= e^{2x^3} \times 6x^2$$

$$= 6x^2 \times e^{2x^3}$$

Example 2:

Differentiate $y = \log_e x^3$:

$\therefore$

$$\frac{dy}{dx} = \frac{1}{x^3} \times \frac{d}{dx}(x^3)$$

$$= \frac{1}{x^3} \times 3x^2$$

$$= \frac{3}{x}$$

We also have, $y = \log_e x^3 = 3 \log_e x$

$$\frac{dy}{dx} = 3 \times \frac{1}{x} = \frac{3}{x}$$

Example 3:

Differentiate $\log_e \frac{x^3}{x^3 - 1}$

$$y = \log_e x^3 - \log_e (x^3 - 1)^{\frac{1}{2}}$$

$$= 3 \log_e x - \frac{1}{2} \log_e (x^3 - 1)$$

$$\frac{dy}{dx} = \frac{3}{x} - \frac{1}{2} \times \frac{1}{x^3 - 1} \times \frac{d}{dx}(x^3 - 1)$$

$$= \frac{3}{x} - \frac{1}{2} \times \frac{1}{x^3 - 1} \times 3x^2$$

$$= \frac{3}{x} - \frac{3x^2}{2(x^3 - 1)}$$

$$\frac{6(x^3 - 1) - 3x^3}{2x(x^3 - 1)}$$

$$\frac{3x^3 - 1}{2x(x^3 - 1)}$$

2.5 Differentiation of the trigonometric functions 1/

When considering differentiation of trigonometric functions, the angle, the function of which is being examined, is assumed to be measured in circular measures; angles will be regarded as measured in radians, often expressed in the convenient form of fractions or multiples of π radians ($\pi = 180°$)

The differential coefficient of a trigonometric function gives the rate of increase of the function with respect to the angle θ (or x expressed in radians).

Function (y)	$\frac{dy}{dx}$
$y = \sin x$	$\cos x$
$y = \cos x$	$-\sin x$
$y = \tan x$	$\sec^2 x$
$y = \operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$y = \sec x$	$\sec x \tan x$
$y = \cot x$	$-\operatorname{cosec}^2 x$

1/ See also: Part III - Trigonometry

Example 1:

Differentiate $y = \sin^3 x$:

$$\frac{dy}{dx} = 3 \sin^2 x \times \frac{d}{dx}(\sin x)$$

$$= 3 \sin^2 x \cos x$$

Example 2:

Differentiate $y = \sin \sqrt{x}$:

$$\frac{dy}{dx} = \cos (x^{\frac{1}{2}}) \times \frac{d}{dx}(x^{\frac{1}{2}})$$

$$\cos (x^{\frac{1}{2}}) \times \frac{1}{2}(x^{-\frac{1}{2}})$$

$$\frac{\cos \sqrt{x}}{2 \sqrt{x}}$$

Example 3:

Differentiate $y = (\cos x^2)^2$:

$$\frac{dy}{dx} = 2 \cos x^2 \times (- \sin x^2) \times 2x$$

$$= -4x \cos x^2 \sin x^2$$

Example 4:

Differentiate $3 \sin x - 4 \cos x$:

Let $y = 3 \sin x - 4 \cos x$

Then $\frac{dy}{dx} = 3 \cos x - 4 (-\sin x)$

$$= 3 \cos x + 4 \sin x$$

2.5.1 Successive derivatives

Let $y = \sin x$

Then
$$\frac{dy}{dx} = \cos x$$
$$\frac{d^2y}{dx^2} = -\sin x$$
$$\frac{d^3y}{dx^3} = -\cos x$$
$$\frac{d^4y}{dx^4} = \sin x$$

These derivatives will be repeated in sets of four, identical to the first four above.

$\therefore$ From the above functions, we have:
$$\frac{dy}{dx} = \cos x$$

But
$$\cos x = \sin (x + \frac{\pi}{2})$$

Substituting, we get:
$$\frac{dy}{dx} = \cos x = \sin (x + \frac{\pi}{2})$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\sin (x + \frac{\pi}{2})\right]$$
$$= \cos (x + \frac{\pi}{2}) = \sin (x + \frac{2\pi}{2})$$
$$\frac{d^3y}{dx^3} = \frac{d}{dx} \sin (x + \frac{2\pi}{2})$$
$$= \cos (x + \frac{2\pi}{2}) = \sin (x + \frac{3\pi}{2})$$

They may be continued indefinitely, $\frac{\pi}{2}$ being added in each successive derivative, the sine form being retained.

Thus, it may be deducted that:
$$\frac{d^ny}{dx^n} = \sin (x + \frac{n\pi}{2})$$

Successive derivatives of $\cos x$ may be similarly obtained.

2.5.2 Maximum and minimum values of trigonometric functions

Example 1:

Find the turning points of the curve $y = \sin x + \cos x$:

$\therefore$ $$\frac{dy}{dx} = \cos x - \sin x$$

$$\frac{dy}{dx} = 0 \quad , \quad \cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$

But this is the smallest of a series of angles whose tangent is +1. All these angles are included in the general formula,

$$n\pi + \frac{\pi}{4}$$

The angles for which there are turning points in the above formula are:

$$\frac{\pi}{4} , \frac{5\pi}{4} , \frac{9\pi}{4} , \ldots$$

Also, $$\frac{d^2y}{dx^2} = -\sin x - \cos x$$

This is negative when, $x = \pi \quad 9\pi$

and positive when, $x = \frac{5\pi}{4} , \frac{13\pi}{4} ,$

$\therefore$ The curve is periodic, and maximum and minimum values occur alternately.

Maximum when, $x = \frac{\pi}{4} , \frac{9\pi}{4} , \ldots$

Minimum when, $x = \frac{5\pi}{4} , \frac{13\pi}{4} , \ldots$

The maximum value of y,

$$y = \sin \frac{\pi}{4} + \cos \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

Similarly, the minimum value $= -\sqrt{2}$

Example 2:

Find the turning points of the curve of $\sin x$:

$$y = \sin x$$

$$\frac{dy}{dx} = \cos x \quad , \quad \cos x = 0$$

$$x = \frac{\pi}{2}$$

This is the smallest of a series of angles, the cosine of which is zero. All these angles are included in the general formula,

$$2n\pi \pm \frac{\pi}{2}$$

The angles for which there are points in the above function are :

$$\frac{\pi}{2} , \frac{3\pi}{2} , \frac{5\pi}{2} , \ldots$$

$$\frac{d^2y}{dx^2} = -\sin x$$

(i) When, $x = \frac{\pi}{2}$, $\frac{d^2y}{dx^2}$ = negative

and y has a maximum value.

(ii) When, $x = \frac{3\pi}{2}$, $\frac{d^2y}{dx^2}$ = positive

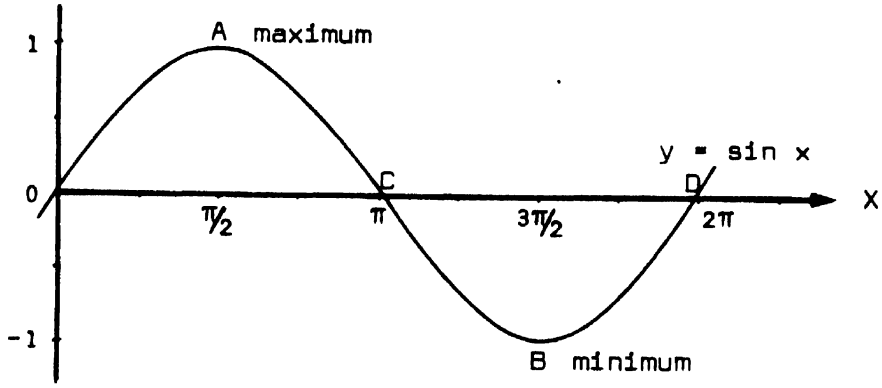
and y has a minimum value.

The above is true for any section of 2π as x increases. Consequently, throughout the curve from $-\infty$ to $+\infty$, there is an infinite sequence of turning points, alternatively maximum and minimum.

$\therefore$ There are two points of inflection on the section 2π of the curve (see Figure below), i.e. points C and D.

At C the curve changes from concave down to concave up, $\frac{dy}{dx}$ is a minimum curve, viz., -1, $\frac{d^2y}{dx^2} = 0$ and changes from negative to positive.

At D this is reversed. The curve changes from concave up to concave down, $\frac{dy}{dx}$ is maximum, viz., $+1$, $\frac{d^2y}{dx^2} = 0$, and the sign changes from positive to negative.



2.5.3 Inverse circular functions (differentiation)

When we write $y = \sin x$, the angle is the independent variable and the sine the dependent. But we may require to reverse this relation, i.e. to express the angle as a function of the sine. The sine now becomes the independent variable and the angle the dependent. This relation is expressed by the form:

$$y = \sin^{-1} x$$

which means, y is an angle of which x is the sine. From this meaning we can write down the direct relation, viz:

$$x = \sin y$$

Differential coefficients of inverse functions

Function (y)	dy/dx
$\sin^{-1} x$	$\frac{1}{\sqrt{1 - x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1 - x^2}}$
$\tan^{-1} x$	$\frac{1}{1 + x^2}$
$\cot^{-1} x$	$-\frac{1}{1 + x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2 - 1}}$
$\operatorname{cosec}^{-1} x$	$-\frac{1}{x\sqrt{x^2 - 1}}$

2.5.4 Hyperbolic functions (differentiation)

1. Differential coefficients of hyperbolic functions

Function (y)	$\frac{dy}{dx}$
$y = \sin h\, x$	$\cos h\, x$
$y = \cos h\, x$	$\sin h\, x$
$y = \tan h\, x$	$\sec h^2 x$
$y = \operatorname{cosec} h\, x$	$-\operatorname{cosec} h\, x \quad \cot h\, x$
$y = \sec h\, x$	$-\sec h\, x \quad \tan h\, x$
$y = \cot h\, x$	$-\operatorname{cosec} h^2 x$

2. Differential coefficients of inverse functions

Function (y)	$\frac{dy}{dx}$
$\sin h^{-1} x$	$\frac{1}{\sqrt{x^2 + 1}}$
$\cos h^{-1} x$	$\frac{1}{\sqrt{x^2 - 1}}$
$\tan h^{-1} x$	$\frac{1}{1 - x^2}$
$\operatorname{cosec} h^{-1} x$	$\frac{-1}{x\sqrt{1 + x^2}}$
$\sec h^{-1} x$	$-\frac{1}{x\sqrt{1 - x^2}}$
$\cot h^{-1} x$	$-\frac{1}{x^2 - 1}$

2.6 Integration, indefinite integrals

In differentiation, we are given, say y = (some function of x), and it is required to find $\frac{dy}{dx}$. In the inverse process, called integration, we are given $\frac{dy}{dx}$ and are required to find y:

Integration of certain functions

Integral	Result
$\int 0dx$	c
$\int dx$	$x + c$
$\int kdx$, (k = constant)	$kx + c$
$\int xdx$	$\frac{x^2}{2} + c$
$\int x^n dx$, ($n \neq 1$	$\frac{x^{n+1}}{n+1} + c$
$\int \frac{dx}{x}$	$\log_e x + c$ (or $\ln x + c$)
$\int e^x dx$	$e^x + c$
$\int a^x dx$	$\frac{a^x}{\ln a} + c$
$\int \cos x \, dx$	$\sin x + c$
$\int \sin x \, dx$	$-\cos x + c$
$\int \frac{dx}{\cos^2 x}$	$\tan x + c$
$\int \frac{dx}{\sin^2 x}$	$-\cot x + c$
$\int \frac{dx}{\sqrt{1 - x^2}}$	$\arcsin x + c$
$\int -\frac{dx}{\sqrt{1 - x^2}}$	$\arccos x + c$
$\int \frac{dx}{1 - x^2}$	$\arctan x + c$

The operation of integration necessitates a symbol to indicate it. The one chosen is $\int$. If $\psi = \phi(x)$, $\int \phi(x)\,dx$ means that $\phi(x)$ is to be integrated with respect to x.

2.6.1 Methods of integration

(a) Integration of a sum

The integral of a sum of a number of functions is equal to the sum of the integrals of these functions.

Examples: Find the following integrals,

(1) $$\int (2x^3 - 5x^2 + 7x - 11)dx$$

$$= 2\int x^3dx - 5\int x^2dx + 7\int x\,dx - 11\int dx$$

$$= \frac{2x^4}{4} - \frac{5x^3}{3} + \frac{7x^2}{2} - 11x + c$$

(2) $$\int (\sqrt{x} - \frac{1}{\sqrt{x}})\,dx$$

$$= \int x^{\frac{1}{2}}dx - \int x^{-\frac{1}{2}}dx$$

$$\frac{x^{\frac{1}{2}+1}}{3/2} - \frac{x^{-\frac{1}{2}+1}}{1/2} + c$$

$$\frac{2x^{3/2}}{3} - 2x^{\frac{1}{2}} + c$$

$$\frac{2x\sqrt{x}}{3} - 2\sqrt{x} + c$$

(b) Integration of fractional functions [1/]

If $y = \ln [f(x)]$,

$$\frac{dy}{dx} = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$$

Consequently,

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$$

1/ Note: $f'(x) = \frac{d}{dx} f(x)$

Examples: Find the following integrals,

(1) $$\int \frac{dx}{ax} = \frac{1}{a}\int \frac{a}{ax}\,dx, \qquad a = (ax)'$$

$$= \frac{1}{a}\ln ax + c$$

(2) $$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{a}{ax+b}\,dx, \qquad a = (ax+b)'$$

$$= \frac{1}{a}\ln (ax+b) + c$$

(c) Integration by parts

This method of integration is derived from the rule for differentiation of a product of two functions, $u = f(x)$, $v = \phi(x)$:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating throughout with respect to x, we get:

$$uv \qquad \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

Since u and v are functions of x, this may be written more conveniently in the form,

$$uv = \int u\,dv + \int v\,du$$

Thus, if either of the integrals on the right side is known, the other can be found, i.e.,

$$\int u\,dv = uv - \int v\,du$$

Examples: Find the following integrals,

(1) $$\int \ln x\,dx \quad , \quad u = \ln x \quad , \quad v = x$$

$$= x\ln x - \int x\,d\ln x$$

$$= x\ln x - \int x\frac{dx}{x}$$

$$= x\ln x - \int dx = x\ln x - x + c$$
$$= x(\ln x - 1) + c$$

(2) $\int x \cos x \, dx$

$= \int x \, d \sin x \quad , \quad u = x \quad , \quad v = \sin x$

$= x \sin x - \int \sin x \, dx$

$= x \sin x + \int d \cos x$

$= x \sin x + \cos x + c$

(3) $\int xe^{ax} dx$

$\frac{1}{a} \int xe^{ax} d\, ax$

$\frac{1}{a} \int x de^{ax} \quad , \quad u = x \quad , \quad v = e^{ax}$

$\frac{1}{a} \left[xe^{ax} - \int e^{ax} dx \right]$

$\frac{1}{a} \left[xe^{ax} - \frac{1}{a} e^{ax} \right] + c$

$\frac{e^{ax}}{a} (x - \frac{1}{a}) + c$

(d) Integration by substitution

It is sometimes possible, by changing the independent variable, to transform a function into another which can be readily integrated.

Examples: Find the following integrals,

(1) $\int \frac{(1 + x^{1/3})^3 dx}{x}$

By putting $x = y^6$, we get $dx = 6y^5 dy$

We also have $x^{\frac{1}{2}} = y^3$ and $x^{1/3} = y^2$

Substituting in the original integral, we get:

$$\int \frac{(1 + y^2)^3 6y^5 dy}{y^3}$$

$$= 6 \int (1 + y^2)^3 y^2 dy$$

$$= 6 \left[\int (1 + 3y^2 + 3y^4 + y^6) y^2 dy \right]$$

$$= 6 \left[\int (y^2 + 3y^4 + 3y^6 + y^8) dy \right]$$

$$= 6 \left[\int y^2 dy + 3 \int y^4 dy + 3 \int y^6 dx + \int y^8 dy \right]$$

$$= 6 \left[\frac{y^3}{3} + \frac{3y^5}{5} + \frac{3y^7}{7} + \frac{y^9}{9} \right] + c$$

Substituting $x = y^6$, we get:

$$= 6 \left[\frac{x^{1/2}}{3} + \frac{3x^{5/6}}{5} + \frac{3x^{7/6}}{7} + \frac{x^{3/2}}{9} \right] + c$$

(2) $$\int \frac{e^{2x} dx}{1 + e^{2x}}$$

$$= \frac{1}{2} \int \frac{e^{2x} d(2x)}{1 + 2e^{2x}} = \frac{1}{4} \int \frac{d(e^{2x})}{1 + 2e^2}$$

Putting $y = 1 + 2e^{2x}$, $dy = d(2e^{2x})$
Substituting, we get:

$$\frac{1}{4} \int \frac{dy}{y} = \frac{1}{4} \ln y + c$$

$$\ln y^{1/4} + \ln c$$

$$\ln c \sqrt[4]{1 + 2e^{2x}}$$

(3) $\int e^{\sin x} \sin x \cos x \, dx$

Putting $y = \sin x$, $dy = \cos x \, dx$.
Substituting, we get:

$$= \int e^y y \, dy$$

$$= \int y \, de^y$$

$$= ye^y - \int e^y dy$$

$$= ye^y - e^y + c$$

Substituting $y = \sin x$, we get:

$$= \sin x \, e^{\sin x} - e^{\sin x} + c$$

$$e^{\sin x}(\sin x - 1) + c$$

(4) $\int \frac{x dx}{(ax + b)^{3/2}}$

Putting $y^2 = ax + b$, we have $x = \frac{y^2 - b}{a}$, $dx = \frac{2y \, dy}{a}$

Also, $(ax + b)^{3/2} = y^3$

Substituting, we get:

$$= \int \frac{(y^2 - b)2y \, dy}{a \cdot a \cdot y^3}$$

$$= \frac{2}{a^2} \int \frac{(y^2 - b)dy}{y^2}$$

$$= \frac{2}{a^2} \left[\int dy - b \int \frac{dy}{y^2} \right]$$

$$= \frac{2}{a^2} \left[y + \frac{b}{y} \right] + c$$

Substituting $v^2 = ax + b$, we get:

$$= \frac{2}{a^2} \left[\sqrt{ax + b} + \frac{b}{\sqrt{ax + b}} \right] + c$$

2.6.2 Integration of algebraic fractions

2.6.2.1 Method of partial fractions

This method of integration is used when the denominator of the fraction to be integrated can be resolved into linear factors which are different.

Example 1: Integrate,

(1) $$\int \frac{x\,dx}{x^2 - 3x + 2}$$

Factorising the denominator,

(2) $$\frac{x}{x^2 - 3x + 2} = \frac{x}{(x - 1)(x - 2)}$$

Thus, the original fraction is resolvable into two partial fractions with denominators (x-1) and (x-2). Since the numerator of the given fraction is of lower dimensions than the denominator, it is evident that the numerators of the partial fractions will be numbers, i.e. not containing x .

Let the numbers be A and B, so that:

(3) $$\frac{x}{x^2 - 3x + 2} = \frac{A}{x - 1} + \frac{B}{x - 2}$$

Clearing the fractions,

(4) $$x = A(x - 2) + B(x - 1)$$

This is an identity and therefore true for any value of x. Let $x = 1$, by which means the coefficient of B vanishes. Then, $1 = A(1 - 2)$, $A = -1$. Let $x = 2$, by which the coefficient A in (4) vanishes. Then, $2 = B(2 - 1)$, $B = 2$.

Substituting for A and B in (3),

$$\frac{x}{x^2 - 3x + 2} = -\frac{1}{x - 1} + \frac{2}{x - 2}$$

Hence,

$$\int \frac{x\,dx}{x^2 - 3x + 2} = -\int \frac{dx}{x-1} + 2\int \frac{dx}{x-2}$$

$$= -\ln(x-1) + 2\ln(x-2) + \ln c$$

$$= \ln \frac{c(x-2)^2}{x-1}$$

Example 2: Integrate,

$$\int \frac{x+35}{x^2-25}\,dx$$

$\therefore$ $$\frac{x+35}{x^2-25} = \frac{x+35}{(x-5)(x+5)}$$

$$\frac{x+35}{(x-5)(x+5)} = \frac{A}{(x-5)} + \frac{B}{x+5}$$

$$x + 35 = A(x+5) + B(x-5)$$

$\therefore$ Let $x = 5$,

$$5 + 35 = A(5+5), \quad A = 4$$

Let $x = -5$,

$$-5 + 35 = B(-5-5), \quad B = -3$$

$$\frac{x+35}{(x-5)(x+5)} = \frac{4}{x-5} - \frac{3}{x+5}$$

$\therefore$ $$\int \frac{x+35}{(x-5)(x+5)} = 4\int \frac{dx}{x-5} - 3\int \frac{dx}{x+5}$$

$$= 4\ln(x-5) - 3\ln(x+5) + \ln c$$

$$= \ln \frac{c(x-5)^4}{(x+5)^3}$$

2.6.2.2 The numerator of the same or higher dimensions of the denominator

Example 1: Integrate,

$$\int \frac{x^2}{x+1}\,dx \quad :$$

$$\therefore \int \frac{x^2}{x+1}\,dx = \int \frac{(x^2-1)+1}{x+1}\,dx$$

$$= \int \frac{(x^2-1)}{x+1}\,dx + \int \frac{dx}{x+1}$$

$$= \int (x-1)\,dx + \int \frac{dx}{x+1}$$

$$= \frac{1}{2}x^2 - x + \ln(x+1) + c$$

Example 2: Integrate,

$$\int \frac{x^2+10x+6}{x^2+2x-8}\,dx:$$

$$\therefore \frac{x^2+10x+6}{x^2+2x-8} = \frac{(x^2+2x-8)+(8x+14)}{x^2+2x-8}$$

$$= 1 + \frac{8x+14}{x^2+2x-8}$$

$$\therefore \int \frac{x^2+10x+6}{x^2+2x-8} = \int dx + \int \frac{8x+14}{x^2+2x-8}$$

$$\therefore \frac{8x+14}{x^2+2x-8} = \frac{A}{x-2} + \frac{B}{x+4}$$

$$= \frac{5}{x-2} + \frac{3}{x+4}$$

$$\int \frac{8x+14}{x^2+2x-8} = 5\int \frac{dx}{x-2} + 3\int \frac{dx}{x+4}$$

$$= 5\ln(x-2) + 3\ln(x+4) + c$$

$$\therefore x + 5\ln(x-2) + 3\ln(x+4) + c$$

2.6.2.3 The denominator is the square of a binomial

If the denominator is of the form $(x+a)^2$, the fraction may be the sum of two fractions, of which the denominators are $(x+a)$ and $(x+a)^2$, with constants as numerators.

Example 1: Integrate,

(1) $$\int \frac{3x+1}{(x+1)^2}\,dx$$

(2) $$\frac{3x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

(3) $$3x+1 = A(x+1) + B$$

Let $x = -1$. Then $-3+1 = B$, $B = -2$

A may be found by using the property of identity, viz., the coefficients of like terms on the two sides of the identity are equal. Comparing the coefficients of x in (3), we get $A = 3$.

$\therefore$ $$\int \frac{3x+1}{(x+1)^2} = 3\int \frac{dx}{x+1} - 2\int \frac{dx}{(x+1)^2}$$

$$= 3\ln(x+1) + \frac{2}{x+1}$$

Note: $\int \frac{du}{u^2} = -\frac{1}{u}$, $(u = x+1)$.

Example 2: Integrate,

$$\int \frac{(x-1)dx}{(x+1)(x^2+1)}$$

$\therefore$ $$\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

(1) $x - 1 = A(x^2 + 1) + (Bx + C)(x + 1)$

Let $x = -1$, $-2 = A(2)$, $A = -1$

Substituting A in (1), we get:

$x - 1 = -(x^2 + 1) + (Bx + C)(x + 1)$

$x^2 + x = (Bx + C)(x + 1)$

Equating coefficients of x^2, $B = 1$

Equating coefficients of x, $1 = B + C$, $C = 0$

Hence,

$$\int \frac{(x - 1)dx}{(x + 1)(x^2 + 1)} = -\int \frac{dx}{x + 1} + \int \frac{x\,dx}{(x^2 + 1)}$$

$$= -\ln(x+1) + \frac{1}{2}\ln(x^2 + 1) + \ln c$$

$$= \ln \frac{c\sqrt{(x^2 + 1)}}{x + 1}$$

2.6.2.4 Integration by tables

Many integrals have been computed and catalogued in so-called tables of integrals. When an integral does not readily yield to any of the three-standard methods,

1. Integration by parts
2. Integration by transformations or substitutions
3. Integration by partial fractions

then it may be possible, by the use of these methods to throw the given integral into a form that can be found in a table of integrals.

2.6.3 Definite integrals

Suppose we wish to find that function y of x whose derivative is $f'(x)$ and whose value is y_0 when $x = x_0$, we seek $y = \int f'(x)dx$, satisfying the condition $y = y_0$ when $x = x_0$. Let the indefinite integral of $f'(x)$ be $f(x) + c$; that is,

(1) $$\int f'(x)dx = f(x) + c$$

To impose the condition $y = y_0$ when $x = x_0$, means that:

(2) $$y_0 = f(x_0) + c$$

And hence c is determined:

(3) $$c = y_0 - f(x_0)$$

Thus, the solution to the problem is:

(4) $$y = f(x) + y_0 - f(x_0)$$

This is reduced to y_0 when $x = x_0$. The function y in (4) is therefore definite (there is no longer any arbitrary constant in y) and satisfies the initial or boundary condition $y = y_0$ when $x = x_0$ as it is called. Of course y still depends upon x.

Write (4) in the form,

(5) $$y - y_0 = f(x) - f(x_0)$$

And call $y = y_1$, when $x = x_1$. Then (5) becomes:

$$y_1 - y_0 = f(x_1) - f(x_0)$$

$$\int \left[f'(x)dx \right]_{x=x_1} - \int \left[f'(x)dx \right]_{x=x_0}$$

$$\int_{x_0}^{x_1} f'(x)dx$$

This difference is called the definite integral of $f'(x)$; it is constant and does not depend upon x at all.

This difference is also read "the integral from x_0 to x_1 of $f'(x)$"; x_1 is called the upper limit, x_0 the lower limit. Or again x_0 and x_1 are called the limits of integration, and the function $f'(x)$ is integrated between the limits x_0 and x_1.

Example 1: Find $\int_0^1 (x^2 + e^x)dx$

$$\therefore \int_0^1 (x^2 + e^x)dx = \left[\frac{x^3}{3} + e^x \right]_0^1$$

$$= \left[\frac{1}{3} + e^1 \right] - \left[\frac{0}{3} + e^0 \right] = e - \frac{2}{3}$$

Example 2: Find $\int_0^{\pi} \sin x \, dx$

$$\therefore \int_0^{\pi} \sin x \, dx = \left[- \cos x \right]_0^{\pi} = 1 + 1 = 2$$

Note: (1) $$\int_a^b f'(x)\,dx = -\int_b^a f'(x)\,dx$$

(2) $$\int_a^c f'(x)\,dx = \int_a^b f'(x)\,dx + \int_b^c f'(x)\,dx$$

2.6.3.1 Improper integrals

The question arises as to what meaning is to be attached to a definite integral when one (or both) of the limits of integration is infinite or when the integral itself becomes infinite at or between the limits of integration.

Example 1: Find $\int_0^{\infty} e^{-3x}dx$

$$\therefore \lim_{b\to\infty} \int_0^b e^{-3x}dx = \lim_{b\to\infty} \left[- \frac{1}{3} e^{-3x} \right]_0^b$$

$$= \lim_{b\to\infty} \left[- \frac{1}{3} e^{-3b} + \frac{1}{3} \right] = \frac{1}{3}$$

Example 2: Find $\int_{-1}^{1} \frac{dx}{x^{2/3}}$

$\therefore$ The function becomes infinite at the origin.

Hence,
$$\int_{-1}^{1} \frac{dx}{x^{2/3}} = \lim_{\varepsilon_1 \to 0} \int_{-1}^{-\varepsilon_1} \frac{dx}{x^{2/3}} + \lim_{\varepsilon_2 \to 0} \int_{\varepsilon_2}^{1} \frac{dx}{x^{2/3}}$$

$$= \lim_{\varepsilon_1 \to 0} 3x^{1/3} \Big]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0} 3x^{1/3} \Big]_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon_1 \to 0} 3 \left[(-\varepsilon_1)^{1/3} - (-1)^{1/3} \right] + \lim_{\varepsilon_2 \to 0} 3 (1 - \varepsilon_2^{1/3})$$

$$= 3 + 3 = 6$$

Example 3: Find $\int_{1}^{2} \frac{dx}{(x-1)^2}$

The integral becomes infinite at $x = 1$.

$\therefore$
$$\lim_{\varepsilon \to 0} \int_{1+\varepsilon}^{2} \frac{dx}{(x-1)^2} = \lim_{\varepsilon \to 0} \left[-\frac{1}{(x-1)} \right]_{1+\varepsilon}^{2}$$

$$= \lim_{\varepsilon \to 0} \left[-1 + \frac{1}{\varepsilon} \right]$$

$$= \infty$$

2.7 Partial differentiation

So far we have been dealing only with functions of one independent variable. The existence of two independent variables gives rise to two differential coefficients.

If z is a function of x and y, the partial coefficients are written:

(1) $\frac{\partial z}{\partial x}$ when x is variable, and y constant

(2) $\frac{\partial z}{\partial y}$ when y is variable, and x constant

Example 1: $z = 3x^3 + 4x^2y + xy^2 + 2y^3$

$$\frac{\partial z}{\partial x} = 9x^2 + 8xy + y^2 \quad \text{(y constant)}$$

$$\frac{\partial z}{\partial y} = 4x^2 + 2xy + 6y^2 \quad \text{(x constant)}$$

Example 2: $z = \sin y + x^2 \cos y + e^{2x}$

$$\frac{\partial z}{\partial x} = 2x \cos y + 2e^{2x} \quad \text{(y constant)}$$

$$\frac{\partial z}{\partial x} = \cos y - x^2 \sin y \quad \text{(x constant)}$$

2.7.1 Higher partial derivations

The partial derivatives are themselves functions of the variables concerned, and thus may have their partial derivatives.

Notation:

(1) $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

(2) $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial z}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

(3) $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

(4) $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

2.7.2 Total differential

When a function of a single variable such as $y = f(x)$ is differentiated, the result is expressed by:

(1) $$\frac{dy}{dx} = f'(x)$$

and

(2) $$dy = f'(x)dx$$

Equation (2) expresses the differential of the dependent variable (dy) in terms of the differential of the independent variable (dx).

∴ When $z = f(x,y)$, the total differential of z, where z is a function of the variables x,y and is expressed by:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

A similar expression may be obtained when z is a function of three variables.

2.7.3 Total differential coefficient

Given $z = x^2y + e^{xy}$, $x = t^2$, $y = \ln t$, find $\frac{dz}{dt}$.

∴ $\frac{dz}{dt}$ is called the total differential coefficient, and is given [1/] by:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + ye^{xy})2t + (x^2 + xe^{xy})\frac{1}{t}$$

1/ If y is a function of x, the total differential coefficient of dz is found by replacing t by x in the above equation:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

2.7.4 Implicit functions

Let $z = f(x,y) =$ a constant, say c.

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = 0$$

$$\frac{\partial z}{\partial x}\,dx = -\frac{\partial z}{\partial y}\,dy$$

$$\frac{dy}{dx} = -\left(\frac{\partial z}{\partial x} \div \frac{\partial z}{\partial y}\right)$$

Example: If $z = 2x^3 - xy^2 + y^4 = 0$, find $\frac{dy}{dx}$.

$\therefore$ $$\frac{dy}{dx} = -\left(\frac{\partial z}{\partial x} \div \frac{\partial z}{\partial y}\right)$$

$$\frac{\partial z}{\partial x} = 6x^2 - y^2$$

$$\frac{\partial z}{\partial y} = -2xy + 4y^3$$

$\therefore$ Substituting

$$\frac{dy}{dx} = -\left(\frac{6x^2 - y^2}{-2xy + 4y^3}\right)$$

$$= \frac{6x^2 - y^2}{2xy - 4y^3}$$

2.8 Series

Maclaurin's theorem (or Stirling's theorem)

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \cdots$$

In this form $f^n(0)$ means that in the n^{th} differential coefficient of $f(x)$, x is replaced by 0.

Example 1: Expand $f(x) = e^x$

$\therefore$ $$(e^x)' = e^x \quad , \quad e^0 = 1$$
$$(e^x)'' = e^x \quad , \quad e^0 = 1$$
$$\ldots\ldots$$

$\therefore$ $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \cdots$$

Example 2: Expand $f(x) = \ln(1 + x)$

$$f(x) = (1 + x) \quad , \qquad f(0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{1 + x} \quad , \qquad f'(0) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{(1 + x)^2} \quad , \qquad f''(0) = -\frac{1}{1} = -1$$

$$f'''(x) = \frac{1.2}{(1 + x)^3} \quad , \qquad f'''(0) = 1.2 = 1.2$$

$$\ldots$$

$$\ln(1+x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1} x^n + \ldots$$

GREEK ALPHABET

Letters		Names	Letters		Names
A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ε	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Y	υ	Upsilon
I	$\imath$	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

PART 3 - TRIGONOMETRY

1.0 Introduction

Trigonometry was originally developed in connection with the relationship between the sides and angles of a triangle. Its modern uses are widespread in many theoretical and applied fields. The trigonometric functions force themselves on you in a very surprising fashion when you study the calculus of certain algebraic functions (see Part 2 - Calculus). The complete set consisting of the algebraic functions, the exponential functions, and the trigonometric functions is called the set of "elementary functions". The present part discusses in a summary form the trigonometry of the plane.

1.1 Angles

An angle is formed when two straight lines or two surfaces meet.

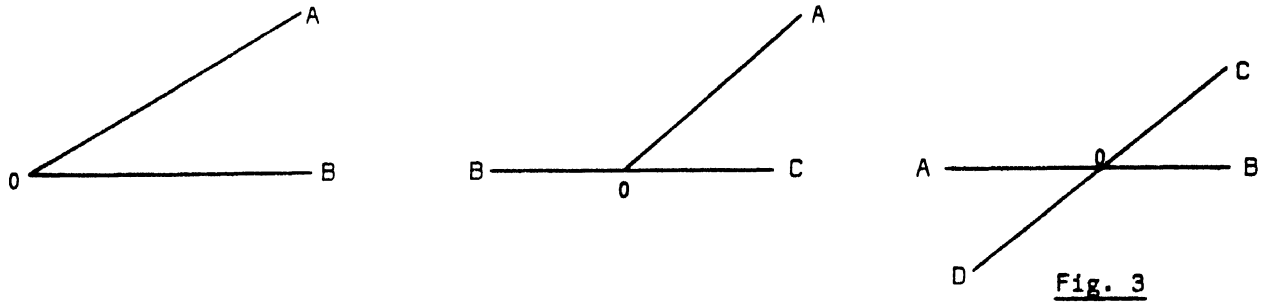


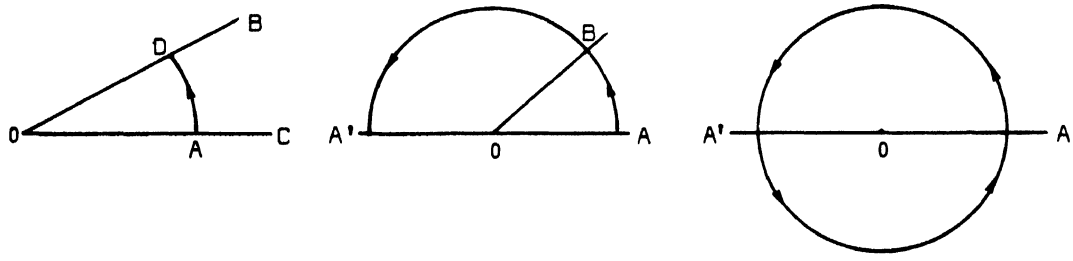
Fig. 3

In Figure 1, the angle is denoted by AOB or <AOB or $A\hat{O}B$.
O is termed the vertex of the angle.
The arms may be of any length, and the size of the angle is not altered by increasing or decreasing them.

In Figure 2, the angles AOB and AOC have a common vertex and a common arm, and are called adjacent angles.

In Figure 3, the pair of angles COB, AOD and COA, BOD are called vertically opposite.

∴ Angles can also be formed by the rotation of a straight line about a fixed point, called the vertex of the angle [1].



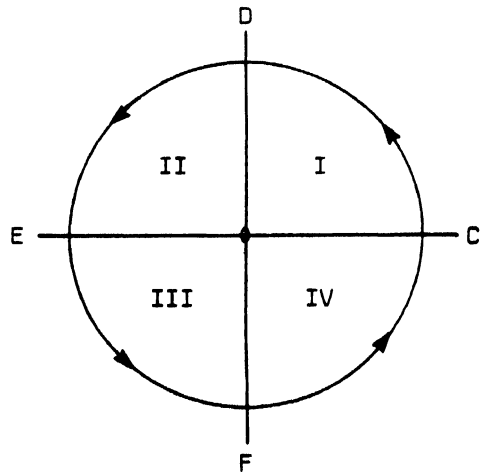
There is no limit to the amount of rotation of the straight line, and consequently angles of any size can be formed by a straight line rotating in this way.

1.2 Measurement of angles

(a) Sexagesimal measure: The formation of an angle by rotation leads to a convenient method of measuring angles. We imagine the complete rotation to be divided into 360 small equal angles, each of which is called a degree, and is denoted by 1°.

Each degree is divided into 60 minutes, shown by '
Each minute is divided into 60 seconds, shown by "
Thus, 30°40'55" means an angle [2] of 30 degrees, 40 minutes, 55 seconds

It should be noted, in the figure below, that the circumference has been divided into four equal parts, here called quadrants. The angles COD, DOE, EOF, FOC, each one fourth of a complete rotation, are termed right angles, and each contains 90°.



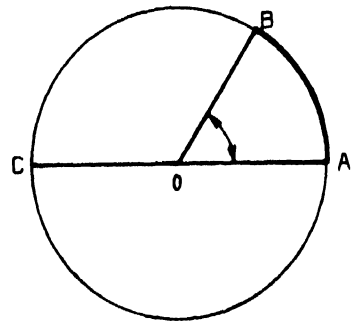
1/ An angle θ might be positive (counter clockwise) or negative (clockwise)

2/ In practice the "circular protractor" is used for the purpose of measuring angles

(b) Centesimal measure: To make the system of measuring angles consistent with other metric measures, it was decided to divide the right angle into 100 equal parts, and consequently the whole cycle into 400 parts or grades.

1 right angle = 100 grades
1 grade = 100 minutes
1 minute = 100 seconds

(c) Circular measure: This measure is an absolute one, that is, it does not depend upon dividing the circle into any arbitrary number of equal parts, such as 360 or 400. The unit is obtained as follows: in a circle, centre O (see figure below), let a radius OA rotate to position OB, such that the length of the arc AB is equal to that of the radius. In doing this, an angle AOB is formed which is the unit of measurement. It is called a radian (= 57°17'44.8", approximately).



1.3 Arc length and measure (circular measure)

The ratio of the circumference of a circle to its diameter is a fixed one for all circles.

$$\therefore \frac{\text{Circumference}}{\text{diameter}} = \pi \left(= \frac{22}{7} = 3.14159 \ldots\right)$$

$c = \pi \times$ diameter , c = circumference
$c = 2\pi r$, r = radius

By definition, a radian is the angle subtended at the centre of a circle by an arc equal in length to the radius. Since the angles at the centre of a circle are proportional to the arcs on which they stand, the semi-circle subtends two right angles.

2 right angles = π radians
180° = π radians

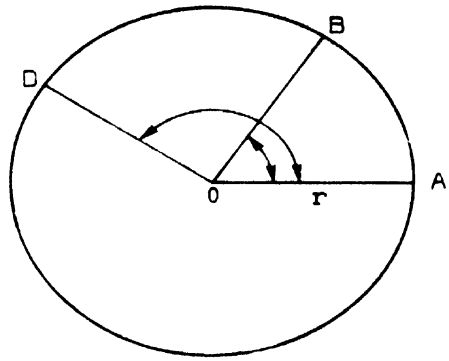
$$\therefore \quad 1 \text{ radian} = \frac{180}{\pi} = 57°17'45'' \text{ approximately}$$

The circular measure of any angle:

$\therefore$ In the circle, the length of arc AB = r
The number of radians in $A\hat{O}D$ is:

$$\frac{A\hat{O}D}{A\hat{O}B} = \frac{\text{arc AD}}{\text{arc AB}}$$

If θ = number of radians in $A\hat{O}D$, then



To convert degress to radians:

Since $180° = \pi$ radians

$$1° = \frac{\pi}{180} \text{ radians}$$

and,

$$\therefore \theta° = \left(\theta^0 \times \frac{\pi}{180}\right) \text{radians}$$

To find the length and measure of an arc:

Let a = length of arc
θ = number of radians in angle

Then, as shown above:

$$\frac{\text{length arc}}{\text{radius}} = \text{number of radians in the angle the arc subtends}$$

$$\therefore \quad \theta = \frac{a}{r}$$

$$\text{and} \quad a = r\theta$$

In accordance with the use of π, angles are frequently expressed as multiples or fractions of it:

$$2\pi = 360°$$

$$\pi = 180°$$

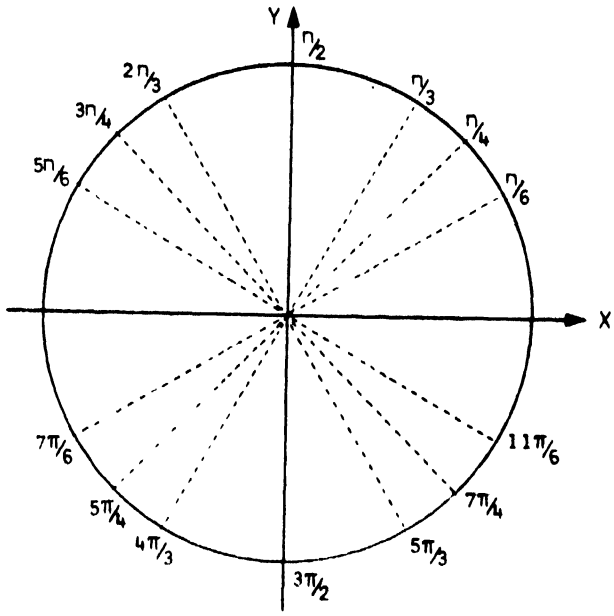
$$\frac{\pi}{2} = 90°$$

$$\frac{\pi}{4} = 45°$$

$$\frac{\pi}{3} = 60°$$

$$\frac{\pi}{6} = 30° \qquad \textit{etc.}$$

∴ Positions of important arcs are given in the graph below.

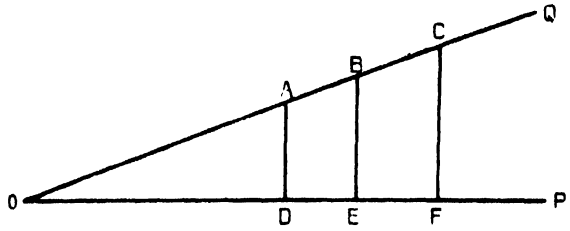


1.4 Tangent

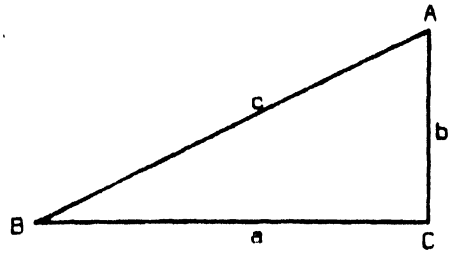
Consider the angle POQ (figure below). From points A,B,C of one arm, draw perpendiculars AD, BE, CF on the other arm. From the produced similar triangles, the following equation is valid:

$$\frac{AD}{OD} = \frac{BE}{OE} = \frac{CF}{OF}$$

For the angle POQ the ratio of the perpendicular drawn from one point to one arm of the angle to the distance intercepted on the other arm is constant. This constant ratio is called the tangent of the angle.



∴ Let ABC be a right-angled triangle. Let the sides opposite the angles be denoted by: a (opposite A), b (opposite B), c (opposite C),



From the above analysis, we have:

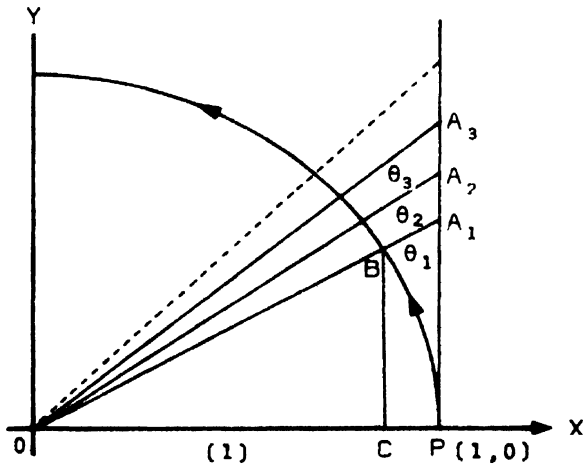
$$\tan B = \frac{AC}{BC} = \frac{b}{a}$$

We also have:

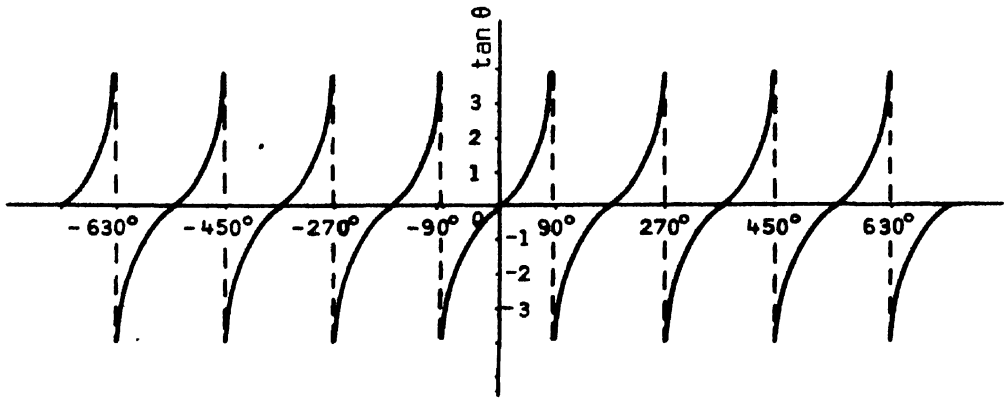
$$b = a \tan B$$

1.4.1 The graph of the function tan θ

We draw a unit circle. Consider several arcs such that $0 < \theta_1 < \theta_2 \ldots < \theta_3 < \pi/2$. Now we erect a line tangent to the circle at P(1,0); we also draw the several lines from the origin to the end points $\theta_1, \theta_2, \theta_3, \ldots$ extending these to intercept the tangent line at $A_1, A_2, A_3, \ldots$ respectively. It is seen that the length of the segment of the tangent PA_1 is actually equal to $\tan \theta_1$, by proportional parts of the triangles OPA_1 and OCB, etc.



The periodicity of the tangent is displayed in the graph below. From this graph one can see that it is periodic with a period of 180°, i.e. two angles which differ by a multiple of 180° must have the same tangent.



Graph of tan θ

The variations of $\tan \theta$ are given in the table below.

Quadrant	As θ varies from:			$\tan \theta$ varies from:		
I	0	to	$\pi/2$	0	to	∞
II	$\pi/2$	to	π	$-\infty$	to	0
III	π	to	$3\pi/2$	0	to	∞
IV	$3\pi/2$	to	2π	$-\infty$	to	0

1.4.2 Table of tangents 1/

Values of a tangent have been tabulated for various degrees of accuracy. In a normal table, the first column indicates the angle in degrees; if the angle includes minutes we must use the remaining columns. For example, by using the tabulated values, we get: $\tan 25°24' = 0.4748$

∴ Examples of the uses of tangents:

1. When the altitude of the sun is $\theta = 48°24'$, find the height of a flagstaff whose shadow is 26 m long:

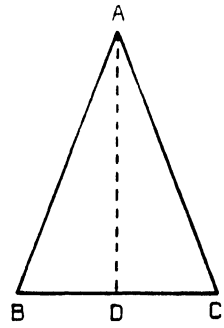
$$\frac{y}{x} = \tan \theta$$

$$y = x \tan \theta$$
$$= 26 \tan 48°24'$$
$$= 26 \times 1.1263$$
$$= 29.28 \text{ m}$$

26 ft

1/ See: Appendix

2. The base of an isosceles triangle is 10 cm, and each of the equal sides is 13 cm. Find the angles of the triangle:



$\therefore$ AB = AC = 13 cm

BD = DC = $\frac{10}{2}$ = 5 cm

$A\hat{B}C = A\hat{C}B$, $B\hat{A}C = 180° - 2A\hat{B}C$

AD = $\sqrt{13^2 - 5^2}$ = 12

$\therefore \tan \hat{B} = \frac{12}{5} = 2.4$

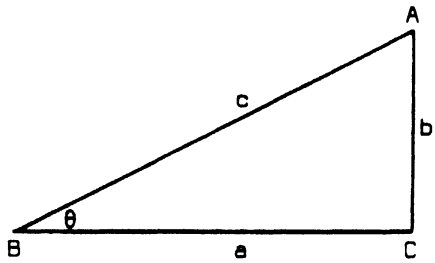
$\hat{B} = 67°24'$

$\hat{C} = 67°24'$

$\hat{A} = 45°12'$

1.5 Sines and Cosines

We have seen that in the case of a right-angled triangle ABC, $\tan \theta = AC/BC$.



Now let us consider the ratios of each of the lines AC and BC to the hypotenuse AB:

(i) The ratio $\frac{AC}{AB}$, i.e. the ratio of the side opposite the angle to the hypotenuse is constant and is called the sine of the angle,

$$\sin \theta = \frac{AC}{AB} = \frac{b}{c}$$

(ii) The ratio $\frac{BC}{AB}$, i.e. the ratio of the intercept to the hypotenuse is also constant for the angle and is called the cosine,

$$\cos \theta = \frac{BC}{AB} = \frac{a}{c}$$

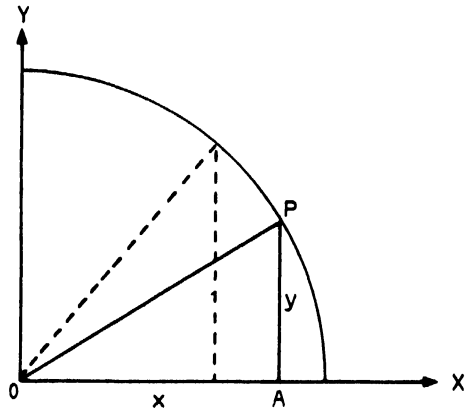
$\sin \theta$ and $\cos \theta$ cannot be greater than unity.

$\therefore$ Note that,

$$\frac{\sin\theta}{\cos\theta} = \frac{AC/AB}{BC/AB} = \frac{AC}{BC} = \tan\theta$$

1.5.1 Changes in the sines and cosines of angles

The figure below shows the rectangular axes OX,OY together with a circle which has centre O and a unit radius.

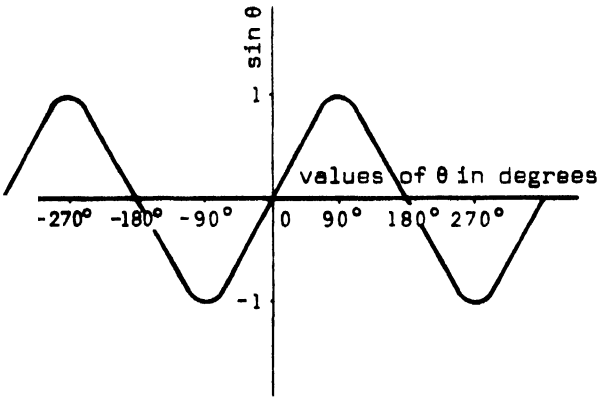


According to our definition, the sine and cosine of any angle θ are the coordinates x,y of the corresponding point (P). The sine and cosine are, like the tangent, functions of the real number θ.

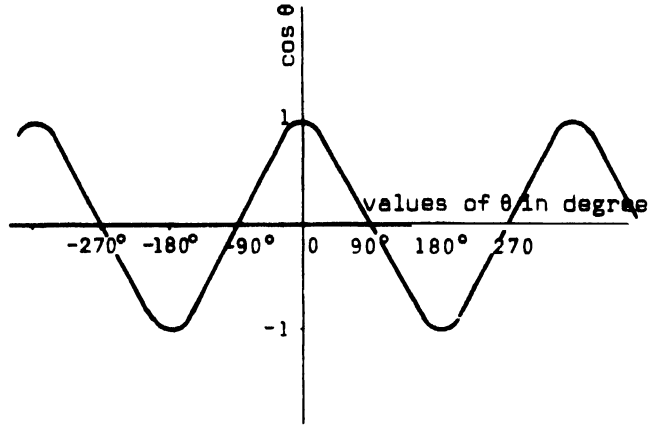
The variations of sine and cosine are given in the table below.

Quadrant	As θ varies from:	sin θ varies from:	cos θ varies from:
I	0 to $\pi/2$	0 to 1	1 to 0
II	$\pi/2$ to π	1 to 0	0 to -1
III	π to $3\pi/2$	0 to -1	-1 to 0
IV	$3\pi/2$ to 2π	-1 to 0	0 to 1

The periodicity of sine and cosine is displayed in the graphs below. Sin θ and cos θ are periodic functions with period 360°, i.e. any two angles that differ by a multiple of 360° must have the same sine and cosine.

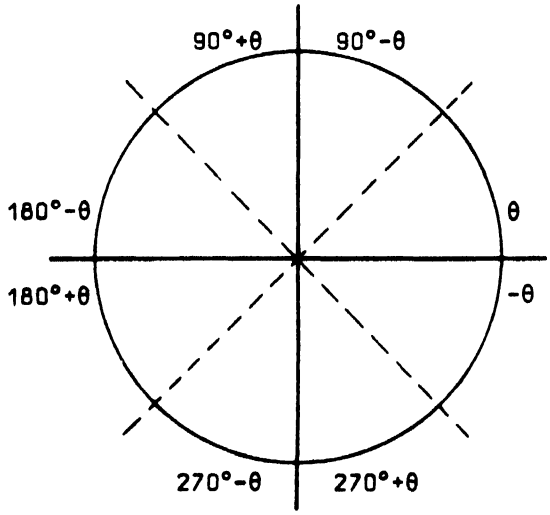


Graph of sin θ



Graph of cos θ

We can obtain a number of useful results simply by looking at the symme of the following figure.



In particular [1/],

$\therefore$ $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos(\theta)$

i.e. sin θ is an odd function, and cos θ is an even function

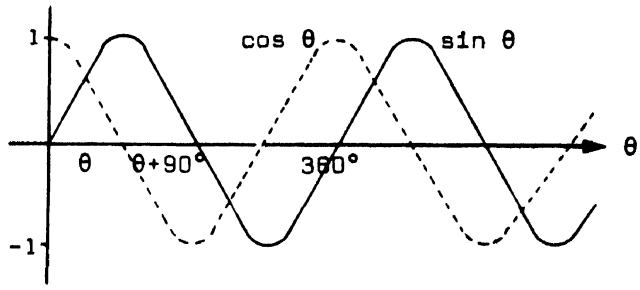
1/ We also have: $\sin\theta = \sin(180° - \theta)$
$\cos\theta = -\cos(180° - \theta)$
$\tan\theta = -\tan(180° - \theta)$

$\therefore$ $\sin(90° - \theta) = +\cos\theta$, $\cos(90° - \theta) = +\sin\theta$

i.e. the $\sin\theta$ of an angle θ is the cosine of its complementary angle $90° - \theta$

$\therefore$ $\sin(90° + \theta) = +\cos\theta$, $\cos(90° + \theta) = -\sin\theta$

Each of these has an interpretation for the graphs of the functions, e.g. $\sin(90° + \theta) = \cos\theta$ shows that the graph of $\sin\theta$ is simply the graph of $\cos\theta$ displaced 90° to the right.



1.5.2 Tables of sines and cosines [1/]

As in the case of tangent ratio, tabulated values of sine and cosine are available and for various degrees of accuracy. These have been calculated and arranged by methods similar to the tangent tables.

The tables for cosines are not really essential when we have the tables of sines, for since $\cos\theta = \sin(90° - \theta)$, we can find cosines of angles from the sine tables.

For example, $\cos\theta = \sin(90° - 47°)$, $(\theta = 47°)$
$= \sin 43°$
$= 0.6820$

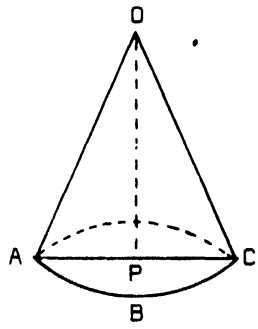
$\therefore$ The sine of an angle is equal to the cosine of its complement, and vice versa. This can be explained in the form,

$\sin\theta = \cos(90° - \theta)$
$\cos\theta = \sin(90° - \theta)$

1/ See: Appendix

Examples of the uses of sines and cosines:

1. The height of a cone is 18 ins, and the angle of the vertex 88°. Find the slant height.



$\therefore$ OP = 18 ins

$A\hat{O}C = 88°$, $P\hat{O}C = 44°$

$\therefore$ OP = OC $\times$ cos $P\hat{O}C$

OC = OP $\div$ cos $P\hat{O}C$
= 18 $\div$ cos 44°
= 18 $\div$ 0.7193
= 25 ins (approx.)

2. The figure here represents a section of a symmetrical roof, PA = 28 ft, AB = 6 ft, <OPA 21°. Find OP and OA.

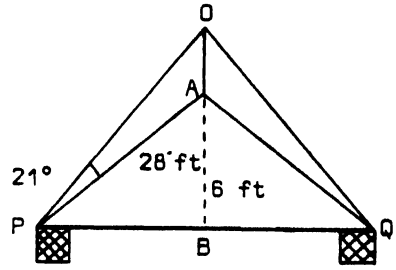
$\therefore$ $O\hat{P}B = O\hat{P}A + A\hat{P}B = 21° + A\hat{P}B$

$\sin A\hat{P}B = \frac{6}{28} = 0.2143$

$A\hat{P}B = 12°23'$

$O\hat{P}B = 21° + 12°23' = 33°23'$

$\therefore$ PB = AP cos $A\hat{P}B$ = 28 $\times$ cos 12°23'
= 28 $\times$ 0.9768
= 27.35 ft



$\therefore$ OP = PB $\div$ cos 33°23'
= 27.35 $\div$ 0.8350
= 32.75 ft

$\therefore$ OA = OB - AB

where,

OB = OP $\times$ sin 33°23'
= 32.75 ft $\times$ 0.5503
= 18.03 ft

OA = 18.03 - 6 = 12.03 ft

1.6 Cosecant, secant and cotangent

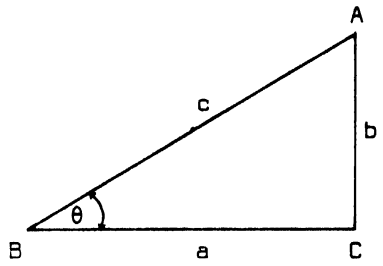
From the reciprocals of sine, cosine and tangent, we can obtain three other ratios connected with an angle:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

These can be expressed in terms of the sides of a right-angled triangle with the usual construction as follows:



$$\sin\theta = \frac{b}{c}, \qquad \operatorname{cosec}\theta = \frac{c}{b}$$

$$\cos\theta = \frac{a}{c}, \qquad \sec\theta = \frac{c}{a}$$

$$\tan\theta = \frac{b}{a}, \qquad \cot\theta = \frac{a}{b}$$

Their properties can be deduced from those of the original functions.

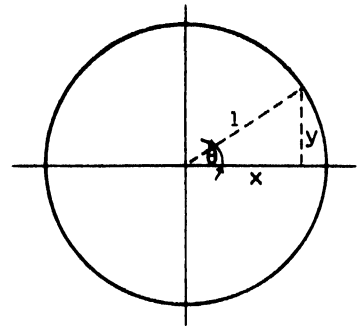
For example,

1. $\tan\theta = \cot(90° - \theta)$, or the tangent of an angle is equal to the cotangent of its complement.

2. Since $\cos\theta = \sin(90° - \theta)$, $\frac{1}{\cos\theta} = \frac{1}{\sin(90° - \theta)}$, so the secant of an angle is the cosecant of the supplementary angle.

1.7 Pythagoras' theorem

It is clear from Pythagoras' theorem that for any angle θ,



$1^2 = y^2 + x^2$, $x = \cos\theta$, $y = \sin\theta$

or,

$1 = (\sin\theta)^2 + (\cos\theta)^2$

or,

$\sin^2\theta + \cos^2\theta = 1$

Dividing both sides of the above equation by $\cos^2\theta$ and $\sin^2\theta$, we get:

$$\therefore \quad \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\text{or,} \quad 1 = \sec^2\theta - \tan^2\theta$$

$$\therefore \quad \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

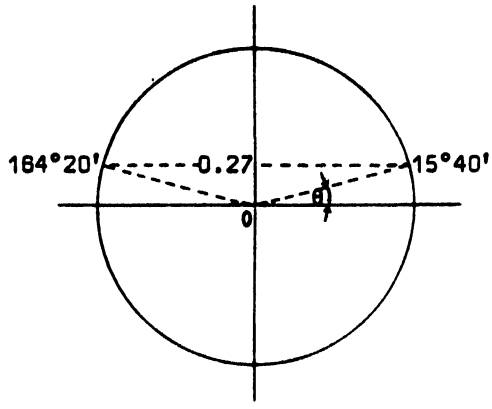
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\text{or,} \quad 1 = \operatorname{cosec}^2\theta - \cot^2\theta$$

Example 1:

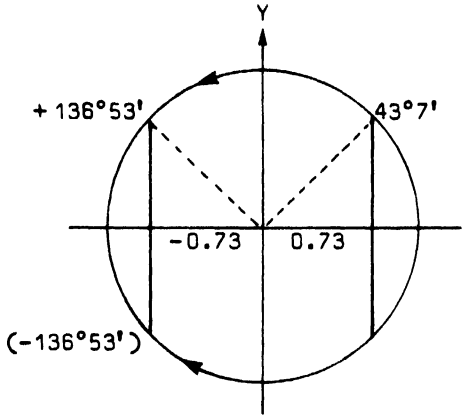
For what values of θ is: (a) $\sin\theta = 0.27$; (b) $\cos\theta = -0.73$; (c) $\tan\theta = -2$

$\therefore$ (a) $\sin\theta = 0.27$



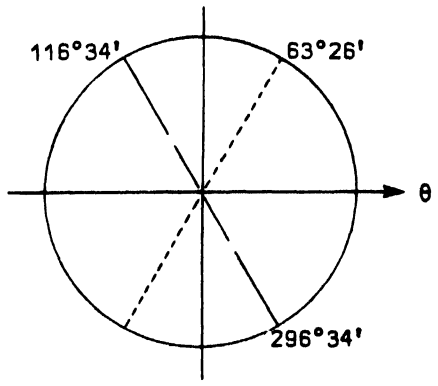
From the tables, we find that $\theta = 15°40'$ is the solution. Hence, from the figure above, it is obvious that another possible solution is $\theta = 164°20'$ ($= 180° - 15°40'$). Also, further solutions can be obtained from these two by adding or subtracting multiples of 360°.

(b) $\cos \theta = -0.73$



From the tables, we find that $\cos \theta = +0.73$ has a solution $\theta = 43°7'$. Hence, from the above figure, we can see that $\cos \theta = -0.73$ has solutions $\pm 136°53'$ ($136°53 = 180° - 43°7'$). Finally, by adding multiples of 360°, we obtain the full set of solutions.

(c) $\tan \theta = -2$



In this case, we look for the line whose gradient is -2. Now $\theta = 63°26'$ for $\tan \theta = +2$. And the first two solutions of $\tan \theta = -2$ are $\theta = 116°34'$ ($= 180° - 63°26'$) and $\theta = 296°34'$ ($= 116°34' + 180°$).

Example 2:

Simplify $\frac{1}{\sec A - \tan A}$:

$$\frac{1}{\sec A - \tan A} = \frac{\sec^2 A - \tan^2 A}{\sec A - \tan A}$$

$$= \frac{(\sec A - \tan A)(\sec A + \tan A)}{\sec A - \tan A}$$

$$= \sec A + \tan A$$

Example 3:

Solve the equation $3 \sin^2 x = 3 \cos^2 x - \cos x + 1$, for values of x between $0°$ and $360°$.

$\therefore$ $\sin^2 x = 1 - \cos^2 x$, substituting in the above equation, we get:

$$3(1 - \cos^2 x) = 3 \cos^2 x - \cos x + 1$$

$$3 - 3 \cos^2 x = 3 \cos^2 x - \cos x + 1$$

$$6 \cos^2 x - \cos x - 2 = 0$$

Putting $y = \cos x$, the above equation can be written:

$$6y^2 - y + 2 = 0$$

$$(3y - 2)(2y + 1) = 0$$

$$y_1 = \frac{2}{3} \quad , \quad y_2 = -\frac{1}{2}$$

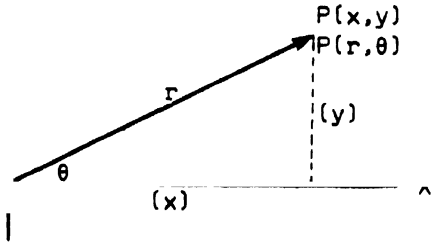
$$\cos x = \frac{2}{3} \quad , \quad \cos x = -\frac{1}{2}$$

$$x = 48°11', 311°49' \text{ or } 120°, 240°$$

So, $x = 48°11', 120°, 240°, 311°49'$

1.8 Polar coordinates

We have seen earlier on how to locate a point on the plane by means of its rectangular coordinates (x,y). For many purposes, however, it is more convenient to locate points by using a different system of coordinates (r,θ), called polar coordinates. In polar coordinates a point is determined by a distance and an angle.



Definitions:

1. The origin O is called the pole.

2. The horizontal (initial) ray, directed to the right from O, is called the polar axis.

3. Given a point P on the plane, the directed segment OP is called the radius vector (terminal ray). Its length, which we assume here to be positive, is denoted by r.

4. A directed angle (positive or negative) with the polar axis as the initial side and the radius vector on its terminal side is denoted by θ (standard position).

5. In the ordered pair (r,θ), r and θ are called polar coordinates of P.

We now have two sets of coordinates for locating a point on the plane:

(a) Rectangular (x,y)
(b) Polar (r,θ)

To use these two interchangeably, we need to be able to go from one to the other at will. We observe at once that $r = \sqrt{x^2 + y^2}$.

Further, the following equations are valid:

$$x = r \cos \theta$$
$$y = r \sin \theta$$

From the above equations one can readily find the rectangular coordinates when the polar coordinates are given.

1.9 Logarithms of trigonometric functions

Calculations in trigonometry are shortened and obtained more accurately by using the logarithms of the function.

Example:

Find the value of the product $\sin 57° \times \tan 24°$

Let, $x = \sin 57° \times \tan 24°$

$= 0.8387 \times 0.4452$

Taking logs, $\log x = \log (0.8387) + \log (0.4452)$

$= \bar{1}.9236 + \bar{1}.6486$

$= \bar{1}.5722$

$x = 0.3734$

Tables are available giving:

Logarithms of sines
Logarithms of cosines
Logarithms of tangents

The logarithms of cosecants, secants and cotangents are the logarithms of their reciprocals, e.g:

$$\sec \theta = \frac{1}{\cos \theta}$$

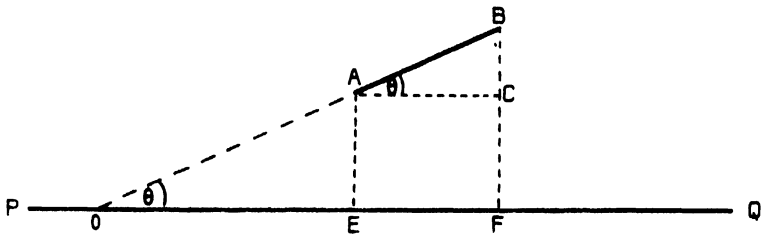
$$\log (\sec \theta) = \log (1) - \log (\cos \theta)$$

$$= 0 - \log (\cos \theta)$$

$$= -\log \cos \theta$$

It should be noted that the logarithm of a number is equal to:
- (log of its reciprocal).

2.0 Projections



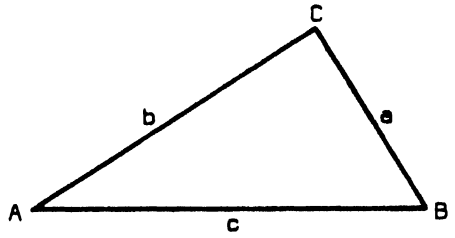
Let PQ be a straight line of unlimited length, and AB another straight line which, when produced to meet PQ at O, makes an angle θ with it.

In the above figure it is indicated that the length of the projection of AB is EF which is equal to:

$$EF = AB \cos \theta$$

2.1 Sine and cosine rules

If we know the sizes a,b,c of the three sides of a triangle ABC, then we can construct it and the angles $\hat{A},\hat{B},\hat{C}$ are automatically determined. The existing relationship between a,b,c and $\hat{A},\hat{B},\hat{C}$ is:



$$\therefore \quad \frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

and,

$$a^2 = b^2 + c^2 - 2bc \cos \hat{A}$$

$$b^2 = a^2 + c^2 - 2ac \cos \hat{B}$$

$$c^2 = a^2 + b^2 - 2ab \cos \hat{C}$$

Example:

If in the triangle ABC, a = 4, b = 3 and c = 2, find the angles $\hat{A},\hat{B},\hat{C}$.

$\therefore$ Using the above cosine rule, we have:

$$4^2 = 3^2 + 2^2 - 2 \times 3 \times 2 \cos \hat{A}$$

$$\cos \hat{A} = -0.25$$

$$\cos 74°31' = +0.25$$

$$\cos (180° - 74°31') = -0.25$$

$$\hat{A} = 104°29'$$

Using the sine rule, we have:

$$\frac{}{\sin 104°29'} = \frac{3}{\sin B} = \frac{2}{\sin C}$$

$$\sin B = \frac{3}{4} \sin 104°29'$$

$$= \frac{3}{4} \times 0.9682 = 0.7261$$

$$46°34'$$

$$\sin C = \frac{2}{4} \sin 104°29' = 0.4841$$

$$28°57'$$

$$A + B + C = (104°29') + (46°34') + (28°57') = 180°$$

2.2 Functions of compound angles

Let us consider a function f and its values $f(x_1)$ and $f(x_2)$. The following question arises: what can we say about $f(x_1 + x_2)$ in terms of $f(x_1)$ and $f(x_2)$ separately? Such a theorem is referred to as an addition theorem for the function f. They are of special importance in trigonometry where we wish to know the answers to related questions.

Addition formulas:

1. $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$

2. $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$

3. $$\tan (A \pm B) = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B}$$

$$= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Example:

Find expressions for the values of sin 75°, cos 75°, tan 75°.

$$\sin 75° = \sin (40° + 35°)$$

$$= \sin 40° \cos 35° + \cos 40° \sin 35°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos 75° = \cos (40° + 35°)$$

$$= \cos 40° \cos 35° - \sin 40° \sin 35°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\tan 75° = \tan (40° + 35°)$$

$$= \frac{\tan 40° + \tan 35°}{1 - \tan 40° \tan 35°}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}}$$

$$= \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

2.3 Multiple angles

1. Double-angle formulas:

$$\sin 2A = 2 \sin A \cos A \quad , \quad \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad , \quad \cos A \text{ 1/} = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$= 1 - 2 \sin^2 A \qquad = 1 - 2 \sin^2 \frac{A}{2}$$

$$= 2 \cos^2 A - 1 \qquad = 2 \cos^2 \frac{A}{2} - 1$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad , \quad \tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

1/ $\cos A = \dfrac{1 - \tan^2 \frac{A}{2}}{\sec^2 \frac{A}{2}}$, $\sec^2 \theta = 1 + \tan^2 \theta$

2. Half-angle formulas:

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$$

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}$$

$$\tan \frac{A}{2} = \frac{\sin (A/2)}{\cos (A/2)}$$

$$= \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

$$= \frac{1 - \cos A}{\sin A}$$

$$= \frac{\sin A}{1 + \cos A}$$

Example:

Find the angles between 0° and 360° which satisfy:

$$\therefore \quad \cos 2\theta = 7 \cos \theta + 3$$

$$2 \cos^2\theta - 1 = 7 \cos \theta + 3$$

$$2 \cos^2\theta - 7 \cos \theta - 4 = 0$$

Putting $\cos \theta = y$, we have:

$$2y^2 - 7y - 4 = 0$$

$$(2y + 1)(y - 4) = 0$$

$$y_1 = -\tfrac{1}{2} \quad , \quad y_2 = 4$$

And, $\cos \theta = -\tfrac{1}{2}$ or 4 (which is impossible)

$$\theta = 120° \text{ or } 240°$$

2.4 Solutions of triangles

By this it is meant that, given certain sides and angles of a triangle, we proceed to find the others:

Case 1: Three sides known (a, b, c)

(a) The cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad , \quad \text{etc.}$$

(b) The half-angle formula:

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad , \quad s = \frac{1}{2}(a+b+c)$$

However, the formulae for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ may be used:

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \quad , \quad \text{etc.}$$

(c) The sine formula:

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \quad , \quad \text{etc.}$$

Case 2: Given two sides and the contained angle.
If the given sides are b,c and the angle A, we have:

(a) The cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A \quad , \quad \text{etc.}$$

(b) the half-angle formula:

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} \quad , \quad \text{etc.}$$

Case 3: Two angles and a side

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$\therefore$ The area of a triangle is equal to half the product of two sides and the sine of the angle contained by them.

$\therefore$ The area of a triangle in terms of the sides is given by:

$$\text{Area} = \sqrt{s\,(s-a)(s-b)(s-c)}$$

2.5 Factor formulae

There is a group of important factor formulae:

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$= 2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{B-A}{2}\right)$$

Example:

Solve, for values of θ between $0°$ and $360°$, the equation:

$\therefore$ $\cos x + \cos 2x = \sin 2x - \sin x$

$$2 \cos\frac{3x}{2} \cos\frac{x}{2} = 2 \cos\frac{3x}{2} \sin\frac{x}{2}$$

$$2 \cos\frac{3x}{2} \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) = 0$$

Case 1: $2 \cos\frac{3x}{2} = 0$

$$\cos\frac{3x}{2} = 0$$

$$\frac{3x}{2} = 90°, 270°, 450°, \ldots$$

$$x = 60°, 180°, 300°, \ldots$$

Case 2: $\sin \frac{x}{2} - \cos \frac{x}{2} = 0$

$$\sin \frac{x}{2} = \cos \frac{x}{2}$$

$$\tan \frac{x}{2} = 1$$

$$\frac{x}{2} = 45°, 225°, \ldots$$

$$x = 90°, 450°, \ldots$$

So the required solutions are:

$$x = 60°, 90°, 180°, 300°$$

2.6 Ratios of angles greater than 360°

Let $A\hat{O}P$ be any angle, θ, which has been formed by rotation in an anti-clockwise or positive direction from position OA.

Suppose now the rotating line continues to rotate in the same direction for a complete rotation or 360° from OP so that it arrives in the same position, OP, as before. The total amount of rotation from OA is now $360° + \theta$ or $(2\pi + \theta)$ radians.

Clearly, the trigonometrical ratios of this new angle $2\pi + \theta$ must be the same as θ, so that $\sin(2\pi + \theta) = \sin \theta$ and so on for the other ratios.

Similarly, if OP made a rotation in a clockwise direction, i.e. negative direction, from its position, we should have the angle $-2\pi + \theta$. The trigonometrical ratios of this angle will be the same as those of θ.

In general, the trigonometrical ratios of the angles $2n\pi + \theta$, where n is any integer, positive or negative, are the same as those of θ.

$\therefore$ The general formula for all angles which have a given sine is:

$$n\pi + (-1)^n \theta$$

(The formula for the cosecant will be the same).

Where,

n : is any integer +ve or -ve

θ : is the smallest angle having the given sine

∴ The general formula for all angles with a given cosine 1/ is:

$$2n\pi \pm \theta$$

∴ The general formula for all angles with a given tangent is:

$$n\pi + \theta$$

The same formula stands for the cotangent.

2.7 Ratios of angles of any magnitude

$$\therefore \sin\theta = \sin(\pi - \theta) = -\sin(\pi + \theta) = -\sin(2\pi - \theta) = -\sin(-\theta)$$

$$\therefore \cos\theta = -\cos(\pi - \theta) = -\cos(\pi + \theta) = \cos(2\pi - \theta) = \cos(-\theta)$$

$$\therefore \tan\theta = -\tan(\pi - \theta) = \tan(\pi + \theta) = -\tan(2\pi - \theta) = -\tan(-\theta)$$

Examples:

1. Find the value of tan 392°:

$$\tan 392° = 2\pi + 32° = 32° = 0.6249$$

2. Find the value of sin 245°:

$$-\sin(180 + 65°) = \sin 65°$$
$$\sin(180 + 65°) = -\sin 65° = -0.9063$$

1/ The formula for the secant will be the same

3. Find the value of cos 325°:

$$\cos (360° - 35°) = \cos 35°$$
$$= 0.8192$$

4. Find the value of sec 253°:

$$\sec 253° = -\sec (180° + 73°)$$
$$= -\sec 73°$$
$$= -3.4203$$

2.8 Oscillations: amplitude, period and frequency

Sine and cosine functions are used to describe all kinds of waves, vibrations and oscillations.

Terminology

Sinusoidal wave occurs in innumerable ways and places in astronomy, mathematics, and all the sciences including the social sciences:

Amplitude: is the maximum displacement of the oscillation

Period : the time required for a single cycle of events which are regularly repeated

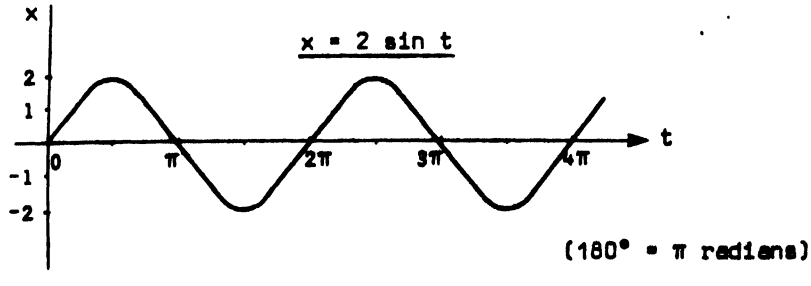
Frequency: is the number of oscillations (or cycles) per second 1/

Example:

Two objects A and B, are bobbing up and down in the water, and after t seconds, their heights xm above their equilibrium positions, are:

(a) $x = 2 \sin t$ (b) $x = 3 \sin 5\pi t$

∴ Object A: $x = 2 \sin t$



1/ In electrical situations, 1 cycle per second is usually called 1 Hertz: $1\ s^{-1} = 1$ Hz

In the above graph we plotted x against t.

The maximum displacement (amplitude) of the oscillation is: $x = 2$m

The period of the oscillation is: $p = 2\pi = 6.28$ s

The frequency of the oscillation (see footnote on p.161) is:

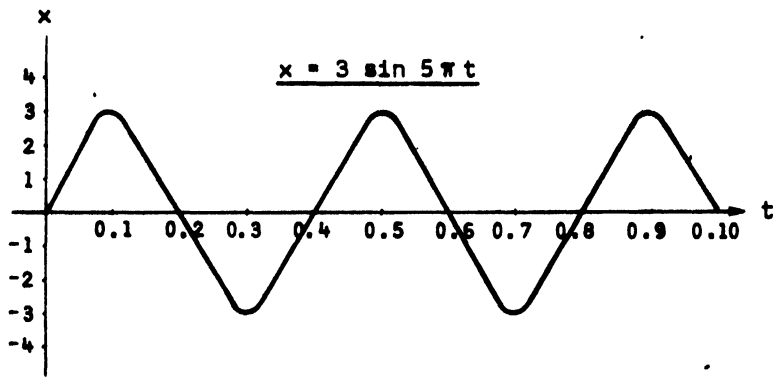
$$f = \frac{1}{2\pi} \approx \frac{1}{6.28} \approx 0.159 \text{ s}^{-1}$$

The above equation $x = 2 \sin t$ enables us to find the displacement at any moment of time; after 5 seconds, for instance:

$$\begin{aligned} x &= 2 \sin 5 \qquad (5 \text{ radians} = 286°30') \\ &= 2 \sin 286°30' \\ &= 2 \times (-0.9588) = -1.92 \end{aligned}$$

So the object is 1.92 m below its equilibrium position (and so only 0.08 m above its lowest point).

$\therefore$ (b) Object B: $x = 3 \sin 5\pi t$



In this case the graph has:

(i) a larger amplitude, 3 m

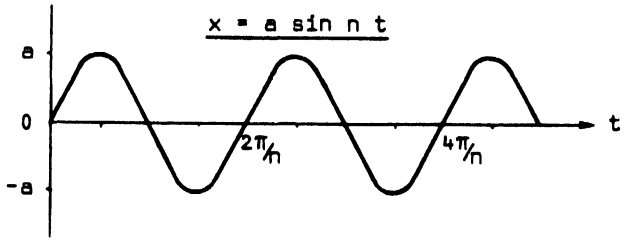
(ii) successive cycles beginning when

$5\pi t$	0	2π	4π	6π	...
t	0	0.4	0.8	1.2	...

Hence, the period $p = 0.4$ s

Frequency $= \frac{1}{0.4} = 2.5 \text{ s}^{-1}$

$\therefore$ More generally, $x = a \sin nt$



has,

(i) amplitude a

(ii) successive cycles beginning at

nt	0	2π	4π	6π	...
t	0	$\frac{2\pi}{n}$	$\frac{4\pi}{n}$	$\frac{6\pi}{n}$	...

so its period is $\frac{2\pi}{n}$

(iii) its frequency is $1 \Big/ \frac{2\pi}{n} = \frac{n}{2\pi}$

2.9 Trigonometrical equations

Trigonometrical equations are those in which the unknown quantities, whose values we require, are the trigonometrical ratios of the angles. The angles themselves can be determined when the values of the ratios are known.

There are practically no general rules which, if followed, will lead to the roots of a trigonometric equation. You might try to factor or to solve by quadratic formula where appropriate. Or, again, you might reduce each and every trigonometric function present to one and the same function of one and the same independent variable. In this section, we exhibit some of the obvious ways of solving such an equation.

Example 1:

Solve the equation,

$$2 \cos \theta = 0.842$$
$$\cos \theta = 0.421$$
$$\theta = 65°6'$$

$\therefore$ The angles between $0°$ and $360°$ which satisfy the equation are:

$$\cos\theta = \cos(2\pi - \theta)$$
$$= 360° - 65°6' = 294°54'$$

The two solutions are $65°6'$ and $294°54'$.

$\therefore$ A general expression for all angles which satisfy the equation are:

$$2n\pi \pm \theta$$

Example 2:

Solve the equation, $2\sin^2 x + \sin x - 1 = 0$

If $y = \sin x$ the above equation can be written,

$$2y^2 + y - 1 = 0$$
$$(2y - 1)(y + 1) = 0$$
$$y_1 = \tfrac{1}{2}, \; y_2 = -\tfrac{1}{2}$$

and,

$\therefore$
$$\sin x = \tfrac{1}{2}$$
$$x = 30° = \frac{\pi}{6}$$

A general expression for all angles which satisfy the equation is:

$$x = \frac{\pi}{6} + 2n\pi \quad (1^{st} \text{ quadrantal arc})$$
$$x = \frac{5\pi}{6} + 2n\pi \quad (2^{nd} \text{ quadrantal arc})$$

$\therefore$
$$\sin x = -1$$
$$-\sin x = 1 = \sin(2\pi + x)$$
$$x = 270° = \frac{3\pi}{2}$$

A general expression for all angles which satisfy the equation is:

$$x = \frac{3\pi}{2} + 2n\pi$$

Example 3:

Obtain a complete solution of the equation,

$$3 \sin \theta = 2 \cos^2 \theta$$

$\therefore$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$3 \sin \theta = 2 (1 - \sin^2 \theta)$$

$$2 \sin^2 \theta + 3 \sin \theta - 2 = 0$$

Substituting, $y = \sin \theta$, $y^2 = \sin^2 \theta$ we get:

$$2y^2 + 3y - 2 = 0$$

$$(2y - 1)(y + 2) = 0$$

$$y_1 = \frac{1}{2} \quad , \quad y_2 = -2$$

and,

$$\sin \theta = \frac{1}{2} \quad , \quad \sin \theta = -2 \text{ (impossible)}$$

$$\theta = 30^\circ \text{ or } \frac{\pi}{6}$$

The general solution for the equation is:

$$\theta = n\pi + (-1)^n \frac{\pi}{6}$$

Example 4:

Solve the equation,

$$\sin 2x + \sin x = 0$$

$\therefore$

$$\sin 2x = 2 \sin x \cos x$$

$$2 \sin x \cos x + \sin x = 0$$

$$\sin x (2 \cos x + 1) = 0$$

$$\sin x = 0 \quad , \quad \cos x = -\tfrac{1}{2}$$

$$x = n\pi \qquad x = \frac{2\pi}{3} + 2n\pi \text{ (2nd quadrantal arc)}$$

$$x = \frac{4\pi}{3} + 2n\pi \text{ (3rd quadrantal arc)}$$

3.0 Hyperbolic trigonometry

Let P(x,y) be a point on the circle in the first quadrant, and set ϕ = arc AP. The area of the sector OAP is equal (see Fig. 1):

$$\text{Area OAP} = r^2 \frac{\phi}{2} = \frac{\phi}{2} \text{ square units } (r = 1, \text{ radius})$$

Since the area of the triangle OAB is equal:

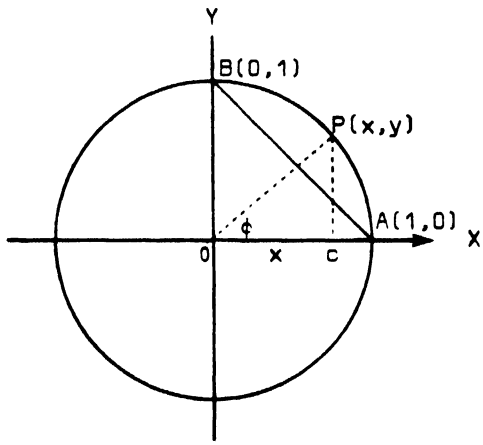
$$\text{Area OAB} = \frac{1 \times 1}{2} = \frac{1}{2}$$

the number ϕ may be thought of as the ratio of the area of the sector to the area of the triangle:

$$\phi = \frac{\text{area of sector OAP}}{\text{area of triangle OAB}} = \frac{\phi/2}{1/2} = \phi$$

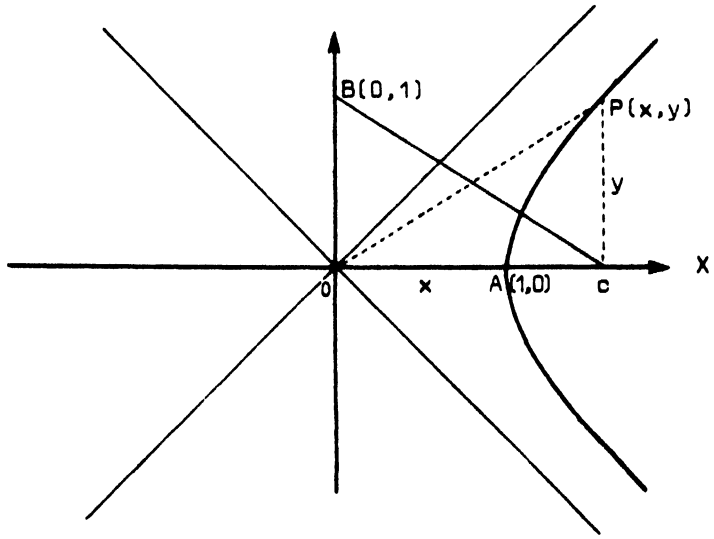
Also, we have: $\sin \phi = y$, $\cos \phi = x$

Fig. 1



By analogy with the above figure (Figure 1), we define the real number θ as twice the area of the hyperbolic sector OAP, where $x^2-y^2=1$ (Fig.2). This θ is not the ordinary measure of the angle AOP; we call it the "hyperbolic measure" of this angle. We now define: $\sin h\ \theta = PC = y$; $\cos h\ \theta = OC = x$

Fig. 2



$$\therefore \quad \theta = \ln (x + y)$$

$$e^{\theta} = x + y \ , \quad e^{-\theta} = \frac{1}{x + y}$$

$$\frac{e^{\theta} - e^{-\theta}}{2} = y = \sin h\, \theta$$

$$\frac{e^{\theta} + e^{-\theta}}{2} = x = \cos h\, \theta$$

$$\therefore \quad \tan h\, \theta = \frac{\sin h\, \theta}{\cos h\, \theta} = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

$$\cot h\, \theta = \frac{\cos h}{\sin h} = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}}$$

$$\sec h\, \theta = \frac{1}{\cos h\, \theta}$$

$$\csc h\, \theta = \frac{1}{\sin h\, \theta} \ , \quad \theta \neq 0$$

A trigonometry of hyperbolic functions can be developed comparable with that of the circular functions.

3.1 Euler's formula

This formula permits us to write $\sin \theta$ and $\cos \theta$ in terms of values of the exponential function in forms very similar to those of the definitions of $\sin h\, \theta$ and $\cos h\, \theta$:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

It follows:

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

A reasonable definition of $e^{i\theta}$ is therefore:

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots)$$

$$= \cos \theta + i \sin \theta$$

APPENDICES

(TABLES)

Table of important constants

π	3.14159
π^2	9.86960
$\sqrt{\pi}$	1.77245
$\frac{1}{\pi}$	0.31830
$\frac{1}{\sqrt{2\pi}}$	0.39894
$\frac{1}{2\pi}$	0.15915
e	2.71828
e^2	7.38905
$\sqrt{e}$	1.64872
$\frac{1}{e}$	0.36787
$\frac{1}{\sqrt{e}}$	0.60653
$\log_e 10$	2.30258
$\log_{10} e$	0.43429

SQUARES, CUBES, ROOTS

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$
1	1	1.000	1	1.000	51	2 601	7.141	132 651	3.708
2	4	1.414	8	1.260	52	2 704	7.211	140 608	3.733
3	9	1.732	27	1.442	53	2 809	7.280	148 877	3.756
4	16	2.000	64	1.587	54	2 916	7.348	157 464	3.780
5	25	2.236	125	1.710	**55**	3 025	7.416	166 375	3.803
6	36	2.449	216	1.817	56	3 136	7.483	175 616	3.826
7	49	2.646	343	1.913	57	3 249	7.550	185 193	3.849
8	64	2.828	512	2.000	58	3 364	7.616	195 112	3.871
9	81	3.000	729	2.080	59	3 481	7.681	205 379	3.893
10	100	3.162	1 000	2.154	**60**	3 600	7.746	216 000	3.915
11	121	3.317	1 331	2.224	61	3 721	7.810	226 981	3.936
12	144	3.464	1 728	2.289	62	3 844	7.874	238 328	3.958
13	169	3.606	2 197	2.351	63	3 969	7.937	250 047	3.979
14	196	3.742	2 744	2.410	64	4 096	8.000	262 144	4.000
15	225	3.873	3 375	2.466	**65**	4 225	8.062	274 625	4.021
16	256	4.000	4 096	2.520	66	4 356	8.124	287 496	4.041
17	289	4.123	4 913	2.571	67	4 489	8.185	300 763	4.062
18	324	4.243	5 832	2.621	68	4 624	8.246	314 432	4.082
19	361	4.359	6 859	2.668	69	4 761	8.307	328 509	4.102
20	400	4.472	8 000	2.714	**70**	4 900	8.367	343 000	4.121
21	441	4.583	9 261	2.759	71	5 041	8.426	357 911	4.141
22	484	4.690	10 648	2.802	72	5 184	8.485	373 248	4.160
23	529	4.796	12 167	2.844	73	5 329	8.544	389 017	4.179
24	576	4.899	13 824	2.884	74	5 476	8.602	405 224	4.198
25	625	5.000	15 625	2.924	**75**	5 625	8.660	421 875	4.217
26	676	5.099	17 576	2.962	76	5 776	8.718	438 976	4.236
27	729	5.196	19 683	3.000	77	5 929	8.775	456 533	4.254
28	784	5.292	21 952	3.037	78	6 084	8.832	474 552	4.273
29	841	5.385	24 389	3.072	79	6 241	8.888	493 039	4.291
30	900	5.477	27 000	3.107	**80**	6 400	8.944	512 000	4.309
31	961	5.568	29 791	3.141	81	6 561	9.000	531 441	4.327
32	1 024	5.657	32 768	3.175	82	6 724	9.055	551 368	4.344
33	1 089	5.745	35 937	3.208	83	6 889	9.110	571 787	4.362
34	1 156	5.831	39 304	3.240	84	7 056	9.165	592 704	4.380
35	1 225	5.916	42 875	3.271	**85**	7 225	9.220	614 125	4.397
36	1 296	6.000	46 656	3.302	86	7 396	9.274	636 056	4.414
37	1 369	6.083	50 653	3.332	87	7 569	9.327	658 503	4.431
38	1 444	6.164	54 872	3.362	88	7 744	9.381	681 472	4.448
39	1 521	6.245	59 319	3.391	89	7 921	9.434	704 969	4.465
40	1 600	6.325	64 000	3.420	**90**	8 100	9.487	729 000	4.481
41	1 681	6.403	68 921	3.448	91	8 281	9.539	753 571	4.498
42	1 764	6.481	74 088	3.476	92	8 464	9.592	778 688	4.514
43	1 849	6.557	79 507	3.503	93	8 649	9.644	804 357	4.531
44	1 936	6.633	85 184	3.530	94	8 836	9.695	830 584	4.547
45	2 025	6.708	91 125	3.557	**95**	9 025	9.747	857 375	4.563
46	2 116	6.782	97 336	3.583	96	9 216	9.798	884 736	4.579
47	2 209	6.856	103 823	3.609	97	9 409	9.849	912 673	4.595
48	2 304	6.928	110 592	3.634	98	9 604	9.899	941 192	4.610
49	2 401	7.000	117 649	3.659	99	9 801	9.950	970 299	4.626
50	2 500	7.071	125 000	3.684	**100**	10 000	10.000	1 000 000	4.642
n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$

VALUES OF e^{x} AND e^{-x}

x	e^{x}	e^{-x}	x	e^{x}	e^{-x}
0.00	1.0000	1.00000	2.10	8.1662	0.12246
0.01	1.0101	0.99005	2.20	9.0250	0.11080
0.02	1.0202	0.98020	2.30	9.9742	0.10026
0.03	1.0305	0.97045	2.40	11.023	0.09072
0.04	1.0408	0.96079	2.50	12.182	0.08208
0.05	1.0513	0.95123	2.60	13.464	0.07427
0.06	1.0618	0.94176	2.70	14.880	0.06721
0.07	1.0725	0.93239	2.80	16.445	0.06081
0.08	1.0833	0.92312	2.90	18.174	0.05502
0.09	1.0942	0.91393	3.00	20.086	0.04979
0.10	1.1052	0.90484	3.10	22.198	0.04505
0.20	1.2214	0.81873	3.20	24.533	0.04076
0.30	1.3499	0.74082	3.30	27.113	0.03688
0.40	1.4918	0.67032	3.40	29.964	0.03337
0.50	1.6487	0.60653	3.50	33.115	0.03020
0.60	1.8221	0.54881	3.60	36.598	0.02732
0.70	2.0138	0.49659	3.70	40.447	0.02472
0.80	2.2255	0.44933	3.80	44.701	0.02237
0.90	2.4596	0.40657	3.90	49.402	0.02024
1.00	2.7183	0.36788	4.00	54.598	0.01832
1.10	3.0042	0.33287	4.10	60.340	0.01657
1.20	3.3201	0.30119	4.20	66.686	0.01500
1.30	3.6693	0.27253	4.30	73.700	0.01357
1.40	4.0552	0.24660	4.40	81.451	0.01228
1.50	4.4817	0.22313	4.50	90.017	0.01111
1.60	4.9530	0.20190	4.60	99.484	0.01005
1.70	5.4739	0.18268	4.70	109.95	0.00910
1.80	6.0496	0.16530	4.80	121.51	0.00823
1.90	6.6859	0.14957	4.90	134.29	0.00745
2.00	7.3891	0.13534	5.00	148.41	0.00674

COMMON LOGARITHMS
(base 10)

N.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6495	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

COMMON LOGARITHMS

(base 10)

N.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	6875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8673	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	6774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	8375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

NATURAL OR NAPERIAN LOGARITHMS
(base e)

To find the natural logarithm of a number which is 1/10 or 10 times a number whose logarithm is given, subtract from or add to the given logarithm the logarithm of 10

A 0.00 - 0.99

N	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0		5.395	6.088	6.493	6.781	7.004	7.187	7.341	7.474	7.592
0.1	7.697	7.793	7.880	7.960	8.034	8.103	8.167	8.228	8.285	8.339
0.2	8.391	8.439	8.486	8.530	8.573	8.614	8.653	8.691	8.727	8.762
0.3	8.796	8.829	8.861	8.891	8.921	8.950	8.978	9.006	9.032	9.058
0.4	9.084	9.108	9.132	9.156	9.179	9.201	9.223	9.245	9.266	9.287
0.5	9.307	9.327	9.346	9.365	9.384	9.402	9.420	9.438	9.455	9.472
0.6	9.489	9.506	9.522	9.538	9.554	9.569	9.584	9.600	9.614	9.629
0.7	9.643	9.658	9.671	9.685	9.699	9.712	9.726	9.739	9.752	9.764
0.8	9.777	9.789	9.802	9.814	9.826	9.837	9.849	9.861	9.872	9.883
0.9	9.895	9.906	9.917	9.927	9.938	9.949	9.959	9.970	9.980	9.990

B 1.00 - 10.09

N	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.0	0.0 0000	0995	1980	2956	3922	5879	5827	6766	7696	8618
1.1	9531	*0436	*1333	*2222	*3103	*3976	*4842	*5700	*6551	*7395
1.2	0.1 8232	9062	9885	*0701	*1511	*2314	*3111	*3902	*4686	*5464
1.3	0.2 6236	7003	7763	8518	9267	*0010	*0748	*1481	*2208	*2930
1.4	0.3 3647	4359	5066	5767	6464	7156	7844	8526	9204	9878
1.5	0.4 0547	1211	1871	2527	3178	3825	4469	5108	5742	6373
1.6	7000	7623	8243	8858	9470	*0078	*0682	*1282	*1879	*2473
1.7	0.5 3063	3649	4232	4812	5389	5962	6531	7098	7661	8222
1.8	8779	9333	9884	*0432	*0977	*1519	*2058	*2594	*3127	*3658
1.9	0.6 4185	4710	5233	5752	6269	6783	7294	7803	8310	8813
2.0	9315	9813	*0310	*0804	*1295	*1784	*2271	*2755	*3237	*3716
2.1	0.7 4194	4669	5142	5612	6081	6547	7011	7473	7932	8390
2.2	8846	9299	9751	*0200	*0648	*1093	*1536	*1978	*2418	*2855
2.3	0.8 3291	3725	4157	4587	5015	5442	5866	6289	6710	7129
2.4	7547	7963	8377	8789	9200	9609	*0016	*0422	*0826	*1228
2.5	0.9 1629	2028	2426	2822	3216	3609	4001	4391	4779	5166
2.6	5551	5935	6317	6698	7078	7456	7833	8208	8582	8954
2.7	9325	9695	*0063	*0430	*0796	*1160	*1523	*1885	*2245	*2604
2.8	1.0 2962	3318	3674	4028	4380	4732	5082	5431	5779	6126
2.9	6471	6815	7158	7500	7841	8181	8519	8856	9192	9527
3.0	9861	*0194	*0526	*0856	*1186	*1514	*1841	*2168	*2493	*2817
3.1	1.1 3140	3462	3783	4103	4422	4740	5057	5373	5688	6002
3.2	6315	6627	6938	7248	7557	7865	8173	8479	8784	9089
3.3	9392	9695	9996	*0297	*0597	*0896	*1194	*1491	*1788	*2083
3.4	1.2 2378	2671	2964	3256	3547	3837	4127	4415	4703	4990
3.5	5276	5562	5846	6130	6413	6695	6976	7257	7536	7815
3.6	8093	8371	8647	8923	9198	9473	9746	*0019	*0291	*0563
3.7	1.3 0833	1103	1372	1641	1909	2176	2442	2708	2972	3237
3.8	3500	3763	4025	4286	4547	4807	5067	5325	5584	5841
3.9	6098	6354	6609	6864	7118	7372	7624	7877	8128	8379
4.0	8629	8879	9128	9377	9624	9872	*0118	*0364	*0610	*0854
4.1	1.4 1099	1342	1585	1828	2070	2311	2552	2792	3031	3270
4.2	3508	3746	3984	4220	4456	4692	4927	5161	5395	5629
4.3	5862	6094	6326	6557	6787	7018	7247	7476	7705	7933
4.4	8160	8387	8614	8840	9065	9290	9515	9739	9962	0185
4.5	1.5 0408	0630	0851	1072	1293	1513	1732	1951	2170	2388
4.6	2606	2823	3039	3256	3471	3687	3902	4116	4330	4543
4.7	4756	4969	5181	5393	5604	5814	6025	6235	6444	6653
4.8	6862	7070	7277	7485	7691	7898	8104	8309	8515	8719
4.9	8924	9127	9331	9534	9737	9939	*0141	*0342	*0543	*0744

NATURAL OR NAPERIAN LOGARITHMS
(base e)

N	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
5.0	1.6 0944	1144	1343	1542	1741	1939	2137	2334	2531	2728
5.1	2924	3120	3315	3511	3705	3900	4094	4287	4481	4673
5.2	4866	5058	5250	5441	5632	5823	6013	6203	6393	6582
5.3	6771	6959	7147	7335	7523	7710	7896	8083	8269	8455
5.4	8640	8825	9010	9194	9378	9562	9745	9928	*0111	*0293
5.5	1.7 0475	0656	0838	1019	1199	1380	1560	1740	1919	2098
5.6	2277	2455	2633	2811	2988	3166	3342	3519	3695	3871
5.7	4047	4222	4397	4572	4746	4920	5094	5267	5440	5613
5.8	5786	5958	6130	6302	6473	6644	6815	6985	7156	7326
5.9	7495	7665	7834	8002	8171	8339	8507	8675	8842	9009
6.0	9176	9342	9509	9675	9840	*0006	*0171	*0336	*0500	*0665
6.1	1.8 0829	0993	1156	1319	1482	1645	1808	1970	2132	2294
6.2	2455	2616	2777	2938	3098	3258	3418	3578	3737	3896
6.3	4055	4214	4372	4530	4688	4845	5003	5160	5317	5473
6.4	5630	5786	5942	6097	6253	6408	6563	6718	6872	7026
6.5	7180	7334	7487	7641	7794	7947	8099	8251	8403	8555
6.6	8707	8858	9010	9160	9311	9462	9612	9762	9912	*0061
6.7	1.9 0211	0360	0509	0658	0806	0954	1102	1250	1398	1545
6.8	1692	1839	1986	2132	2279	2425	2571	2716	2862	3007
6.9	3152	3297	3442	3586	3730	3874	4018	4162	4305	4448
7.0	4591	4734	4876	5019	5161	5303	5445	5586	5727	5869
7.1	6009	6150	6291	6431	6571	6711	6851	6991	7130	7269
7.2	7408	7547	7685	7824	7962	8100	8238	8376	8513	8650
7.3	8787	8924	9061	9198	9334	9470	9606	9742	9877	*0013
7.4	2.0 0148	0283	0418	0553	0687	0821	0956	1089	1223	1357
7.5	1490	1624	1757	1890	2022	2155	2287	2419	2551	2683
7.6	2815	2946	3078	3209	3340	3471	3601	3732	3862	3992
7.7	4122	4252	4381	4511	4640	4769	4898	5027	5156	5284
7.8	5412	5540	5668	5796	5924	6051	6179	6306	6433	6560
7.9	6686	6813	6939	7065	7191	7317	7443	7568	7694	7819
8.0	7944	8069	8194	8318	8443	8567	8691	8815	8939	9063
8.1	9186	9310	9433	9556	9679	9802	9924	*0047	*0169	*0291
8.2	2.1 0413	0535	0657	0779	0900	1021	1142	1263	1384	1505
8.3	1626	1746	1866	1986	2106	2226	2346	2465	2585	2704
8.4	2823	2942	3061	3180	3298	3417	3535	3653	3771	3889
8.5	4007	4124	4242	4359	4476	4593	4710	4827	4943	5060
8.6	5176	5292	5409	5524	5640	5756	5871	5987	6102	6217
8.7	6332	6447	6562	6677	6791	6905	7020	7134	7248	7361
8.8	7475	7589	7702	7816	7929	8042	8155	8267	8380	8493
8.9	8605	7817	8830	8942	9054	9165	9277	9389	9500	9611
9.0	9722	9834	9944	*0055	*0166	*0276	*0387	*0497	*0607	*0717
9.1	2.2 0827	0937	1047	1157	1266	1375	1485	1594	1703	1812
9.2	1920	2029	2138	2246	2354	2462	2570	2678	2786	2894
9.3	3001	3109	3216	3324	3431	3538	3645	3751	3858	3965
9.4	4071	4177	4284	4390	4496	4601	4707	4813	4918	5024
9.5	5129	5234	5339	5444	5549	5654	5759	5863	5968	6072
9.6	6176	6280	6384	6488	6592	6696	6799	6903	7006	7109
9.7	7213	7316	7419	7521	7624	7727	7829	7932	8034	8136
9.8	8238	8340	8442	8544	8646	8747	8849	8950	9051	9152
9.9	9253	9354	9455	9556	9657	9757	3858	9958	*0058	*0158
10.0	2.3 0259	0358	0458	0558	0658	0757	0857	0958	1055	1154

NATURAL SINES

Angle	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	3	6	9	12	15
1°	.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	3	6	9	12	15
2°	.0349	.0366	.0384	.0401	.0419	.0436	.0454	.0471	.0488	.0506	3	6	9	12	15
3°	.0523	.0541	.0558	.0576	.0593	.0610	.0628	.0645	.0663	.0680	3	6	9	12	15
4°	.0698	.0715	.0732	.0750	.0767	.0785	.0802	.0819	.0837	.0854	3	6	9	12	14
5°	.0872	.0889	.0906	.0924	.0941	.0958	.0976	.0993	.1011	.1028	3	6	9	12	14
6°	.1045	.1063	.1080	.1097	.1115	.1132	.1149	.1167	.1184	.1201	3	6	9	12	14
7°	.1219	.1236	.1253	.1271	.1288	.1305	.1323	.1340	.1357	.1374	3	6	9	12	14
8°	.1392	.1409	.1426	.1444	.1461	.1478	.1495	.1513	.1530	.1547	3	6	9	12	14
9°	.1564	.1582	.1599	.1616	.1633	.1650	.1668	.1685	.1702	.1719	3	6	9	12	14
10°	.1736	.1754	.1771	.1788	.1805	.1822	.1840	.1857	.1874	.1891	3	6	9	11	14
11°	.1908	.1925	.1942	.1959	.1977	.1994	.2011	.2028	.2045	.2062	3	6	9	11	14
12°	.2079	.2096	.2113	.2130	.2147	.2164	.2181	.2198	.2215	.2233	3	6	9	11	14
13°	.2250	.2267	.2284	.2300	.2317	.2334	.2351	.2368	.2385	.2402	3	6	8	11	14
14°	.2419	.2436	.2453	.2470	.2487	.2504	.2521	.2538	.2554	.2571	3	6	8	11	14
15°	.2588	.2605	.2622	.2639	.2656	.2672	.2689	.2706	.2723	.2740	3	6	8	11	14
16°	.2756	.2773	.2790	.2807	.2823	.2840	.2857	.2874	.2890	.2907	3	6	8	11	14
17°	.2924	.2940	.2957	.2974	.2990	.3007	.3024	.3040	.3057	.3074	3	6	8	11	14
18°	.3090	.3107	.3123	.3140	.3156	.3173	.3190	.3206	.3223	.3239	3	6	8	11	14
19°	.3256	.3272	.3289	.3305	.3322	.3338	.3355	.3371	.3387	.3404	3	5	8	11	14
20°	.3420	.3437	.3453	.3469	.3486	.3502	.3518	.3535	.3551	.3567	3	5	8	11	14
21°	.3584	.3600	.3616	.3633	.3649	.3665	.3681	.3697	.3714	.3730	3	5	8	11	14
22°	.3746	.3762	.3778	.3795	.3811	.3827	.3843	.3859	.3875	.3891	3	5	8	11	14
23°	.3907	.3923	.3939	.3955	.3971	.3987	.4003	.4019	.4035	.4051	3	5	8	11	14
24°	.4067	.4083	.4099	.4115	.4131	.4147	.4163	.4179	.4195	.4210	3	5	8	11	13
25°	.4226	.4242	.4258	.4274	.4289	.4305	.4321	.4337	.4352	.4368	3	5	8	11	13
26°	.4384	.4399	.4415	.4431	.4446	.4462	.4478	.4493	.4509	.4524	3	5	8	10	13
27°	.4540	.4555	.4571	.4586	.4602	.4617	.4633	.4648	.4664	.4679	3	5	8	10	13
28°	.4695	.4710	.4726	.4741	.4756	.4772	.4787	.4802	.4818	.4833	3	5	8	10	13
29°	.4848	.4863	.4879	.4894	.4909	.4924	.4939	.4955	.4970	.4985	3	5	8	10	13
30°	.5000	.5015	.5030	.5045	.5060	.5075	.5090	.5105	.5120	.5135	3	5	8	10	13
31°	.5150	.5165	.5180	.5195	.5210	.5225	.5240	.5255	.5270	.5284	2	5	7	10	12
32°	.5299	.5314	.5329	.5344	.5358	.5373	.5388	.5402	.5417	.5432	2	5	7	10	12
33°	.5446	.5461	.5476	.5490	.5505	.5519	.5534	.5548	.5563	.5577	2	5	7	10	12
34°	.5592	.5606	.5621	.5635	.5650	.5664	.5678	.5693	.5707	.5721	2	5	7	10	12
35°	.5736	.5750	.5764	.5779	.5793	.5807	.5821	.5835	.5850	.5864	2	5	7	9	12
36°	.5878	.5892	.5906	.5920	.5934	.5948	.5962	.5976	.5990	.6004	2	5	7	9	12
37°	.6018	.6032	.6046	.6060	.6074	.6088	.6101	.6115	.6129	.6143	2	5	7	9	12
38°	.6157	.6170	.6184	.6198	.6211	.6225	.6239	.6252	.6266	.6280	2	5	7	9	11
39°	.6293	.6307	.6320	.6334	.6347	.6361	.6374	.6388	.C401	.6414	2	4	7	9	11
40°	.6428	.6441	.6455	.6468	.6481	.6494	.6508	.6521	.6534	.6547	2	4	7	9	11
41°	.6561	.6574	.6587	.6600	.6613	.6626	.6639	.6652	.6665	.6678	2	4	7	9	11
42°	.6691	.6704	.6717	.6730	.6743	.6756	.6769	.6782	.6794	.6807	2	4	6	9	11
43°	.6820	.6833	.6845	.6858	.6871	.6884	.6896	.6909	.6921	.6934	2	4	6	8	11
44°	.6947	.6959	.6972	.6984	.6997	.7009	.7022	.7034	.7046	.7059	2	4	6	8	10

NATURAL SINES

Angle	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
45°	.7071	.7083	.7096	.7108	.7120	.7133	.7145	.7157	.7169	.7181	2	4	6	8	10
46°	.7193	.7206	.7218	.7230	.7242	.7254	.7266	.7278	.7290	.7302	2	4	6	8	10
47°	.7314	.7325	.7337	.7349	.7361	.7373	.7385	.7396	.7408	.7420	2	4	6	8	10
48°	.7431	.7443	.7455	.7466	.7478	.7490	.7501	.7513	.7524	.7536	2	4	6	8	10
49°	.7547	.7559	.7570	.7581	.7593	.7604	.7615	.7627	.7638	.7649	2	4	6	8	9
50°	.7660	.7672	.7683	.7694	.7705	.7716	.7727	.7738	.7749	.7760	2	4	6	7	9
51°	.7771	.7782	.7793	.7804	.7815	.7826	.7837	.7848	.7859	.7869	2	4	5	7	9
52°	.7880	.7891	.7902	.7912	.7923	.7934	.7944	.7955	.7965	.7976	2	4	5	7	9
53°	.7986	.7997	.8007	.8018	.8028	.8039	.8049	.8059	.8070	.8080	2	3	5	7	9
54°	.8090	.8100	.8111	.8121	.8131	.8141	.8151	.8161	.8171	.8181	2	3	5	7	8
55°	.8192	.8202	.8211	.8221	.8231	.8241	.8251	.8261	.8271	.8281	2	3	5	7	8
56°	.8290	.8300	.8310	.8320	.8329	.8339	.8348	.8358	.8368	.8377	2	3	5	6	8
57°	.8387	.8396	.8406	.8415	.8425	.8434	.8443	.8453	.8462	.8471	2	3	5	6	8
58°	.8480	.8490	.8499	.8508	.8517	.8526	.8536	.8545	.8554	.8563	2	3	5	6	8
59°	.8572	.8581	.8590	.8599	.8607	.8616	.8625	.8634	.8643	.8652	1	3	4	6	7
60°	.8660	.8669	.8678	.8686	.8695	.8704	.8712	.8721	.8729	.8738	1	3	4	6	7
61°	.8746	.8755	.8763	.8771	.8780	.8788	.8796	.8805	.8813	.8821	1	3	4	6	7
62°	.8829	.8838	.8846	.8854	.8862	.8870	.8878	.8886	.8894	.8902	1	3	4	5	7
63°	.8910	.8918	.8926	.8934	.8942	.8949	.8957	.8965	.8973	.8980	1	3	4	5	6
64°	.8988	.8996	.9003	.9011	.9018	.9026	.9033	.9041	.9048	.9056	1	3	4	5	6
65°	.9063	.9070	.9078	.9085	.9092	.9100	.9107	.9114	.9121	.9128	1	2	4	5	6
66°	.9135	.9143	.9150	.9157	.9164	.9171	.9178	.9184	.9191	.9198	1	2	3	5	6
67°	.9205	.9212	.9219	.9225	.9232	.9239	.9245	.9252	.9259	.9265	1	2	3	4	6
68°	.9272	.9278	.9285	.9291	.9298	.9304	.9311	.9317	.9323	.9330	1	2	3	4	5
69°	.9336	.9342	.9348	.9354	.9361	.9367	.9373	.9379	.9385	.9391	1	2	3	4	5
70°	.9397	.9403	.9409	.9415	.9421	.9426	.9432	.9438	.9444	.9449	1	2	3	4	5
71°	.9455	.9461	.9466	.9472	.9478	.9483	.9489	.9494	.9500	.9505	1	2	3	4	5
72°	.9511	.9516	.9521	.9527	.9532	.9537	.9542	.9548	.9553	.9558	1	2	3	3	4
73°	.9563	.9568	.9573	.9578	.9583	.9588	.9593	.9598	.9603	.9608	1	2	2	3	4
74°	.9613	.9617	.9622	.9627	.9632	.9636	.9641	.9646	.9650	.9655	1	2	2	3	4
75°	.9659	.9664	.9668	.9673	.9677	.9681	.9686	.9690	.9694	.9699	1	1	2	3	4
76°	.9703	.9707	.9711	.9715	.9720	.9724	.9728	.9732	.9736	.9740	1	1	2	3	3
77°	.9744	.9748	.9751	.9755	.9759	.9763	.9767	.9770	.9774	.9778	1	1	2	3	3
78°	.9781	.9785	.9789	.9792	.9796	.9799	.9803	.9806	.9810	.9813	1	1	2	2	3
79°	.9816	.9820	.9823	.9826	.9829	.9833	.9836	.9839	.9842	.9845	1	1	2	2	3
80°	.9848	.9851	.9854	.9857	.9860	.9863	.9866	.9869	.9871	.9874	0	1	1	2	2
81°	.9877	.9880	.9882	.9885	.9888	.9890	.9893	.9895	.9898	.9900	0	1	1	2	2
82°	.9903	.9905	.9907	.9910	.9912	.9914	.9917	.9919	.9921	.9923	0	1	1	2	2
83°	.9925	.9928	.9930	.9932	.9934	.9936	.9938	.9940	.9942	.9943	0	1	1	1	2
84°	.9945	.9947	.9949	.9951	.9952	.9954	.9956	.9957	.9959	.9960	0	1	1	1	1
85°	.9962	.9963	.9965	.9966	.9968	.9969	.9971	.9972	.9973	.9974	0	0	1	1	1
86°	.9976	.9977	.9978	.9979	.9980	.9981	.9982	.9983	.9984	.9985	0	0	1	1	1
87°	.9986	.9987	.9988	.9989	.9990	.9990	.9991	.9992	.9993	.9993	0	0	0	1	1
88°	.9994	.9995	.9995	.9996	.9996	.9997	.9997	.9997	.9998	.9998	0	0	0	0	0
89°	.9998	.9999	.9999	.9999	.9999	1.000	1.000	1.000	1.000	1.000	0	0	0	0	0

NATURAL COSINES

Angle	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	1.0000	1.000	1.000	1.000	1.000	1.000	.9999	.9999	.9999	.9999	0	0	0	0	0
1°	.9998	.9998	.9998	.9997	.9997	.9997	.9996	.9996	.9995	.9995	0	0	0	0	0
2°	.9994	.9993	.9993	.9992	.9991	.9990	.9990	.9989	.9988	.9987	0	0	0	0	0
3°	.9986	.9985	.9984	.9983	.9982	.9981	.9980	.9979	.9978	.9977	0	0	1	1	1
4°	.9976	.9974	.9973	.9972	.9971	.9969	.9968	.9966	.9965	.9963	0	0	1	1	1
5°	.9962	.9960	.9959	.9957	.9956	.9954	.9952	.9951	.9949	.9947	0	1	1	1	1
6°	.9945	.9943	.9942	.9940	.9938	.9936	.9934	.9932	.9930	.9928	0	1	1	1	2
7°	.9925	.9923	.9921	.9919	.9917	.9914	.9912	.9910	.9907	.9905	0	1	1	2	2
8°	.9903	.9900	.9898	.9895	.9893	.9890	.9888	.9885	.9882	.9880	0	1	1	2	2
9°	.9877	.9874	.9871	.9869	.9866	.9863	.9860	.9857	.9854	.9851	0	1	1	2	2
10°	.9848	.9845	.9842	.9839	.9836	.9833	.9829	.9826	.9823	.9820	1	1	2	2	3
11°	.9816	.9813	.9810	.9806	.9803	.9799	.9796	.9792	.9789	.9785	1	1	2	2	3
12°	.9781	.9778	.9774	.9770	.9767	.9763	.9759	.9755	.9751	.9748	1	1	2	3	3
13°	.9744	.9740	.9736	.9732	.9728	.9724	.9720	.9715	.9711	.9707	1	1	2	3	3
14°	.9703	.9699	.9694	.9690	.9686	.9681	.9677	.9673	.9668	.9664	1	1	2	3	4
15°	.9659	.9655	.9650	.9646	.9641	.9636	.9632	.9627	.9622	.9617	1	2	2	3	4
16°	.9613	.9608	.9603	.9598	.9593	.9588	.9583	.9578	.9573	.9568	1	2	2	3	4
17°	.9563	.9558	.9553	.9548	.9542	.9537	.9532	.9527	.9521	.9516	1	2	3	4	4
18°	.9511	.9505	.9500	.9494	.9489	.9483	.9478	.9472	.9466	.9461	1	2	3	4	5
19°	.9455	.9449	.9444	.9438	.9432	.9426	.9421	.9415	.9409	.9403	1	2	3	4	5
20°	.9397	.9391	.9385	.9379	.9373	.9367	.9361	.9354	.9348	.9342	1	2	3	4	5
21°	.9336	.9330	.9323	.9317	.9311	.9304	.9298	.9291	.9285	.9278	1	2	3	4	5
22°	.9272	.9265	.9259	.9252	.9245	.9239	.9232	.9225	.9219	.9212	1	2	3	4	6
23°	.9205	.9198	.9191	.9184	.9178	.9171	.9164	.9157	.9150	.9143	1	2	3	5	6
24°	.9135	.9128	.9121	.9114	.9107	.9100	.9092	.9085	.9078	.9070	1	2	4	5	6
25°	.9063	.9056	.9048	.9041	.9033	.9026	.9018	.9011	.9003	.8996	1	3	4	5	6
26°	.8988	.8980	.8973	.8965	.8957	.8949	.8942	.8934	.8926	.8918	1	3	4	5	6
27°	.8910	.8902	.8894	.8886	.8878	.8870	.8862	.8854	.8846	.8838	1	3	4	5	7
28°	.8829	.8821	.8813	.8805	.8796	.8788	.8780	.8771	.8763	.8755	1	3	4	6	7
29°	.8746	.8738	.8729	.8721	.8712	.8704	.8695	.8686	.8678	.8669	1	3	4	6	7
30°	.8660	.8652	.8643	.8634	.8625	.8616	.8607	.8599	.8590	.8581	1	3	4	6	7
31°	.8572	.8563	.8554	.8545	.8536	.8526	.8517	.8508	.8499	.8490	2	3	5	6	8
32°	.8480	.8471	.8462	.8453	.8443	.8434	.8425	.8415	.8406	.8396	2	3	5	6	8
33°	.8387	.8377	.8368	.8358	.8348	.8339	.8329	.8320	.8310	.8300	2	3	5	6	8
34°	.8290	.8281	.8271	.8261	.8251	.8241	.8231	.8221	.8211	.8202	2	3	5	7	8
35°	.8192	.8181	.8171	.8161	.8151	.8141	.8131	.8121	.8111	.8100	2	3	5	7	8
36°	.8090	.8080	.8070	.8059	.8049	.8039	.8028	.8018	.8007	.7997	2	3	5	7	9
37°	.7986	.7976	.7965	.7955	.7944	.7934	.7923	.7912	.7902	.7891	2	4	5	7	9
38°	.7880	.7869	.7859	.7848	.7837	.7826	.7815	.7804	.7793	.7782	2	4	5	7	9
39°	.7771	.7760	.7749	.7738	.7727	.7716	.7705	.7694	.7683	.7672	2	4	6	7	9
40°	.7660	.7649	.7638	.7627	.7615	.7604	.7593	.7581	.7570	.7559	2	4	6	8	9
41°	.7547	.7536	.7524	.7513	.7501	.7490	.7478	.7466	.7455	.7443	2	4	6	8	10
42°	.7431	.7420	.7408	.7396	.7385	.7373	.7361	.7349	.7337	.7325	2	4	6	8	10
43°	.7314	.7302	.7290	.7278	.7266	.7254	.7242	.7230	.7218	.7206	2	4	6	8	10
44°	.7193	.7181	.7169	.7157	.7145	.7133	.7120	.7108	.7096	.7083	2	4	6	8	10

NATURAL COSINES

Angle	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
45°	.7071	.7059	.7046	.7034	.7022	.7009	.6997	.6984	.6972	.6959	2	4	6	8	10
46°	.6947	.6934	.6921	.6909	.6896	.6884	.6871	.6858	.6845	.6833	2	4	6	8	11
47°	.6820	.6807	.6794	.6782	.6769	.6756	.6743	.6730	.6717	.6704	2	4	6	9	11
48°	.6691	.6678	.6665	.6652	.6639	.6626	.6613	.6600	.6587	.6574	2	4	7	9	11
49°	.6561	.6547	.6534	.6521	.6508	.6494	.6481	.6468	.6455	.6441	2	4	7	9	11
50°	.6428	.6414	.6401	.6388	.6374	.6361	.6347	.6334	.6320	.6307	2	4	7	9	11
51°	.6293	.6280	.6266	.6252	.6239	.6225	.6211	.6198	.6184	.6170	2	5	7	9	11
52°	.6157	.6143	.6129	.6115	.6101	.6088	.6074	.6060	.6046	.6032	2	5	7	9	12
53°	.6018	.6004	.5990	.5976	.5962	.5948	.5934	.5920	.5906	.5892	2	5	7	9	12
54°	.5878	.5864	.5850	.5835	.5821	.5807	.5793	.5779	.5764	.5750	2	5	7	9	12
55°	.5736	.5721	.5707	.5693	.5678	.5664	.5650	.5635	.5621	.5606	2	5	7	10	12
56°	.5592	.5577	.5563	.5548	.5534	.5519	.5505	.5490	.5476	.5461	2	5	7	10	12
57°	.5446	.5432	.5417	.5402	.5388	.5373	.5358	.5344	.5329	.5314	2	5	7	10	12
58°	.5299	.5284	.5270	.5255	.5240	.5225	.5210	.5195	.5180	.5165	2	5	7	10	12
59°	.5150	.5135	.5120	.5105	.5090	.5075	.5060	.5045	.5030	.5015	3	5	8	10	13
60°	.5000	.4985	.4970	.4955	.4939	.4924	.4909	.4894	.4879	.4863	3	5	8	10	13
61°	.4848	.4833	.4818	.4802	.4787	.4772	.4756	.4741	.4726	.4710	3	5	8	10	13
62°	.4695	.4679	.4664	.4648	.4633	.4617	.4602	.4586	.4571	.4555	3	5	8	10	13
63°	.4540	.4524	.4509	.4493	.4478	.4462	.4446	.4431	.4415	.4399	3	5	8	10	13
64°	.4384	.4368	.4352	.4337	.4321	.4305	.4289	.4274	.4258	.4242	3	5	8	11	13
65°	.4226	.4210	.4195	.4179	.4163	.4147	.4131	.4115	.4099	.4083	3	5	8	11	13
66°	.4067	.4051	.4035	.4019	.4003	.3987	.3971	.3955	.3939	.3923	3	5	8	11	13
67°	.3907	.3891	.3875	.3859	.3843	.3827	.3811	.3795	.3778	.3762	3	5	8	11	13
68°	.3746	.3730	.3714	.3697	.3681	.3665	.3649	.3633	.3616	.3600	3	5	8	11	14
69°	.3584	.3567	.3551	.3535	.3518	.3502	.3486	.3469	.3453	.3437	3	5	8	11	14
70°	.3420	.3404	.3387	.3371	.3355	.3338	.3322	.3305	.3289	.3272	3	5	8	11	14
71°	.3256	.3239	.3223	.3206	.3190	.3173	.3156	.3140	.3123	.3107	3	6	8	11	14
72°	.3090	.3074	.3057	.3040	.3024	.3007	.2990	.2974	.2957	.2940	3	6	8	11	14
73°	.2924	.2907	.2890	.2874	.2857	.2840	.2823	.2807	.2790	.2773	3	6	8	11	14
74°	.2756	.2740	.2723	.2706	.2689	.2672	.2656	.2639	.2622	.2605	3	6	8	11	14
75°	.2588	.2571	.2554	.2538	.2521	.2504	.2487	.2470	.2453	.2436	3	6	8	11	14
76°	.2419	.2402	.2385	.2368	.2351	.2334	.2317	.2300	.2284	.2267	3	6	8	11	14
77°	.2250	.2233	.2215	.2198	.2181	.2164	.2147	.2130	.2113	.2098	3	6	9	11	14
78°	.2079	.2062	.2045	.2028	.2011	.1994	.1977	.1959	.1942	.1925	3	6	9	11	14
79°	.1908	.1891	.1874	.1857	.1840	.1822	.1805	.1788	.1771	.1754	3	6	9	11	14
80°	.1736	.1719	.1702	.1685	.1668	.1650	.1633	.1616	.1599	.1582	3	6	9	11	14
81°	.1564	.1547	.1530	.1513	.1495	.1478	.1461	.1444	.1426	.1409	3	6	9	12	14
82°	.1392	.1374	.1357	.1340	.1323	.1305	.1288	.1271	.1253	.1236	3	6	9	12	14
83°	.1219	.1201	.1184	.1167	.1149	.1132	.1115	.1097	.1080	.1063	3	6	9	12	14
84°	.1045	.1028	.1011	.0993	.0976	.0958	.0941	.0924	.0908	.0889	3	6	9	12	14
85°	.0872	.0854	.0837	.0819	.0802	.0785	.0767	.0750	.0732	.0715	3	6	9	12	14
86°	.0698	.0680	.0663	.0645	.0628	.0610	.0593	.0576	.0558	.0541	3	6	9	12	15
87°	.0523	.0506	.0488	.0471	.0454	.0436	.0419	.0401	.0384	.0366	3	6	9	12	15
88°	.0349	.0332	.0314	.0297	.0279	.0262	.0244	.0227	.0209	.0192	3	6	9	12	15
89°	.0175	.0157	.0140	.0122	.0105	.0087	.0070	.0052	.0035	.0017	3	6	9	12	15

NATURAL TANGENTS

Angle	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
0°	0.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	3	6	9	12	15
1°	0.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	3	6	9	12	15
2°	0.0349	.0367	.0384	.0402	.0419	.0437	.0454	.0472	.0489	.0507	3	6	9	12	15
3°	0.0524	.0542	.0559	.0577	.0594	.0612	.0629	.0647	.0664	.0682	3	6	9	12	15
4°	0.0699	.0717	.0734	.0752	.0769	.0787	.0805	.0822	.0840	.0857	3	6	9	12	15
5°	0.0875	.0892	.0910	.0928	.0945	.0963	.0981	.0998	.1016	.1033	3	6	9	12	15
6°	0.1051	.1069	.1086	.1104	.1122	.1139	.1157	.1175	.1192	.1210	3	6	9	12	15
7°	0.1228	.1246	.1263	.1281	.1299	.1317	.1334	.1352	.1370	.1388	3	6	9	12	15
8°	0.1405	.1423	.1441	.1459	.1477	.1495	.1512	.1530	.1548	.1566	3	6	9	12	15
9°	0.1584	.1602	.1620	.1638	.1655	.1673	.1691	.1709	.1727	.1745	3	6	9	12	15
10°	0.1763	.1781	.1799	.1817	.1835	.1853	.1871	.1890	.1908	.1926	3	6	9	12	15
11°	0.1944	.1962	.1980	.1998	.2016	.2035	.2053	.2071	.2089	.2107	3	6	9	12	15
12°	0.2126	.2144	.2162	.2180	.2199	.2217	.2235	.2254	.2272	.2290	3	6	9	12	15
13°	0.2309	.2327	.2345	.2364	.2382	.2401	.2419	.2438	.2456	.2475	3	6	9	12	15
14°	0.2493	.2512	.2530	.2549	.2568	.2586	.2605	.2623	.2642	.2661	3	6	9	12	16
15°	0.2679	.2698	.2717	.2736	.2754	.2773	.2792	.2811	.2830	.2849	3	6	9	13	16
16°	0.2867	.2886	.2905	.2924	.2943	.2962	.2981	.3000	.3019	.3038	3	6	9	13	16
17°	0.3057	.3076	.3096	.3115	.3134	.3153	.3172	.3191	.3211	.3230	3	6	10	13	16
18°	0.3249	.3269	.3288	.3307	.3327	.3346	.3365	.3385	.3404	.3424	3	6	10	13	16
19°	0.3443	.3463	.3482	.3502	.3522	.3541	.3561	.3581	.3600	.3620	3	7	10	13	16
20°	0.3640	.3659	.3679	.3699	.3719	.3739	.3759	.3779	.3799	.3819	3	7	10	13	17
21°	0.3839	.3859	.3879	.3899	.3919	.3939	.3959	.3979	.4000	.4020	3	7	10	13	17
22°	0.4040	.4061	.4081	.4101	.4122	.4142	.4163	.4183	.4204	.4224	3	7	10	14	17
23°	0.4245	.4265	.4286	.4307	.4327	.4348	.4369	.4390	.4411	.4431	3	7	10	14	17
24°	0.4452	.4473	.4494	.4515	.4536	.4557	.4578	.4599	.4621	.4642	4	7	11	14	18
25°	0.4663	.4684	.4706	.4727	.4748	.4770	.4791	.4813	.4834	.4856	4	7	11	14	18
26°	0.4877	.4899	.4921	.4942	.4964	.4986	.5008	.5029	.5051	.5073	4	7	11	15	18
27°	0.5095	.5117	.5139	.5161	.5184	.5206	.5228	.5250	.5272	.5295	4	7	11	15	18
28°	0.5317	.5340	.5362	.5384	.5407	.5430	.5452	.5475	.5498	.5520	4	8	11	15	19
29°	0.5543	.5566	.5589	.5612	.5635	.5658	.5681	.5704	.5727	.5750	4	8	12	15	19
30°	0.5774	.5797	.5820	.5844	.5867	.5890	.5914	.5938	.5961	.5985	4	8	12	16	20
31°	0.6009	.6032	.6056	.6080	.6104	.6128	.6152	.6176	.6200	.6224	4	8	12	16	20
32°	0.6249	.6273	.6297	.6322	.6346	.6371	.6395	.6420	.6445	.6469	4	8	12	16	20
33°	0.6494	.6519	.6544	.6569	.6594	.6619	.6644	.6669	.6694	.6720	4	8	13	17	21
34°	0.6745	.6771	.6796	.6822	.6847	.6873	.6899	.6924	.6950	.6976	4	9	13	17	21
35°	0.7002	.7028	.7054	.7080	.7107	.7133	.7159	.7186	.7212	.7239	4	9	13	18	22
36°	0.7265	.7292	.7319	.7346	.7373	.7400	.7427	.7454	.7481	.7508	5	9	14	18	23
37°	0.7536	.7563	.7590	.7618	.7646	.7673	.7701	.7729	.7757	.7785	5	9	14	18	23
38°	0.7813	.7841	.7869	.7898	.7926	.7954	.7983	.8012	.8040	.8069	5	9	14	19	24
39°	0.8098	.8127	.8156	.8185	.8214	.8243	.8273	.8302	.8332	.8361	5	10	15	20	24
40°	0.8391	.8421	.8451	.8481	.8511	.8541	.8571	.8601	.8632	.8662	5	10	15	20	25
41°	0.8693	.8724	.8754	.8785	.8816	.8847	.8878	.8910	.8941	.8972	5	10	16	21	26
42°	0.9004	.9036	.9067	.9099	.9131	.9163	.9195	.9228	.9260	.9293	5	11	16	21	27
43°	0.9325	.9358	.9391	.9424	.9457	.9490	.9523	.9556	.9590	.9623	6	11	17	22	28
44°	0.9657	.9691	.9725	.9759	.9793	.9827	.9861	.9896	.9930	.9965	6	11	17	23	29

NATURAL TANGENTS

Angle	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
45°	1.0000	1.0035	1.0070	1.0105	1.0141	1.0176	1.0212	1.0247	1.0283	1.0319	6	12	18	24	30
46°	1.0355	1.0392	1.0428	1.0464	1.0501	1.0538	1.0575	1.0612	1.0649	1.0686	6	12	18	25	31
47°	1.0724	1.0761	1.0799	1.0837	1.0875	1.0913	1.0951	1.0990	1.1028	1.1067	6	13	19	25	32
48°	1.1106	1.1145	1.1184	1.1224	1.1263	1.1303	1.1343	1.1383	1.1423	1.1463	7	13	20	26	33
49°	1.1504	1.1544	1.1585	1.1626	1.1667	1.1708	1.1750	1.1792	1.1833	1.1875	7	14	21	28	34
50°	1.1918	1.1960	1.2002	1.2045	1.2088	1.2131	1.2174	1.2218	1.2261	1.2305	7	14	22	29	36
51°	1.2349	1.2393	1.2437	1.2482	1.2527	1.2572	1.2617	1.2662	1.2708	1.2753	8	15	23	30	38
52°	1.2799	1.2846	1.2892	1.2938	1.2985	1.3032	1.3079	1.3127	1.3175	1.3222	8	16	24	31	39
53°	1.3270	1.3319	1.3367	1.3416	1.3465	1.3514	1.3564	1.3613	1.3663	1.3713	8	16	25	33	41
54°	1.3764	1.3814	1.3865	1.3916	1.3968	1.4019	1.4071	1.4124	1.4176	1.4229	9	17	26	34	43
55°	1.4281	1.4335	1.4388	1.4442	1.4496	1.4550	1.4605	1.4659	1.4715	1.4770	9	18	27	36	45
56°	1.4826	1.4882	1.4938	1.4994	1.5051	1.5108	1.5166	1.5224	1.5282	1.5340	10	19	29	38	48
57°	1.5399	1.5458	1.5517	1.5577	1.5637	1.5697	1.5757	1.5818	1.5880	1.5941	10	20	30	40	50
58°	1.6003	1.6066	1.6128	1.6191	1.6255	1.6319	1.6383	1.6447	1.6512	1.6577	11	21	32	43	53
59°	1.6643	1.6709	1.6775	1.6842	1.6909	1.6977	1.7045	1.7113	1.7182	1.7251	11	23	34	45	56
60°	1.7321	1.7391	1.7461	1.7532	1.7603	1.7675	1.7747	1.7820	1.7893	1.7966	12	24	36	48	60
61°	1.8040	1.8115	1.8190	1.8265	1.8341	1.8418	1.8495	1.8572	1.8650	1.8728	13	26	38	51	64
62°	1.8807	1.8887	1.8967	1.9047	1.9128	1.9210	1.9292	1.9375	1.9458	1.9542	14	27	41	55	68
63°	1.9626	1.9711	1.9797	1.9883	1.9970	2.0057	2.0145	2.0233	2.0323	2.0413	15	29	44	58	73
64°	2.0503	2.0594	2.0686	2.0778	2.0872	2.0965	2.1060	2.1155	2.1251	2.1348	16	31	47	63	78
65°	2.1445	2.1543	2.1642	2.1742	2.1842	2.1943	2.2045	2.2148	2.2251	2.2355	17	34	51	68	85
66°	2.2460	2.2566	2.2673	2.2781	2.2889	2.2998	2.3109	2.3220	2.3332	2.3445	18	37	55	73	92
67°	2.3559	2.3673	2.3789	2.3906	2.4023	2.4142	2.4262	2.4383	2.4504	2.4627	20	40	60	79	99
68°	2.4751	2.4876	2.5002	2.5129	2.5257	2.5386	2.5517	2.5649	2.5782	2.5916	22	43	65	87	108
69°	2.6051	2.6187	2.6325	2.6464	2.6605	2.6746	2.6889	2.7034	2.7179	2.7326	24	47	71	95	119
70°	2.7475	2.7625	2.7776	2.7929	2.8083	2.8239	2.8397	2.8556	2.8716	2.8878	26	52	78	104	130
71°	2.9042	2.9208	2.9375	2.9544	2.9714	2.9887	3.0061	3.0237	3.0415	3.0595	29	58	87	116	144
72°	3.0777	3.0961	3.1146	3.1334	3.1524	3.1716	3.1910	3.2106	3.2305	3.2506	32	64	96	129	161
73°	3.2709	3.2914	3.3122	3.3332	3.3544	3.3759	3.3977	3.4197	3.4420	3.4646	36	72	108	144	180
74°	3.4874	3.5105	3.5339	3.5576	3.5816	3.6059	3.6305	3.6554	3.6806	3.7062	41	81	122	163	204
75°	3.7321	3.7583	3.7848	3.8118	3.8391	3.8667	3.8947	3.9232	3.9520	3.9812					
76°	4.0108	4.0408	4.0713	4.1022	4.1335	4.1653	4.1976	4.2303	4.2635	4.2972					
77°	4.3315	4.3662	4.4015	4.4374	4.4737	4.5107	4.5483	4.5864	4.6252	4.6646					
78°	4.7046	4.7453	4.7867	4.8288	4.8716	4.9152	4.9594	5.0045	5.0504	5.0970					
79°	5.1446	5.1929	5.2422	5.2924	5.3435	5.3955	5.4486	5.5026	5.5578	5.6140					
80°	5.6713	5.7297	5.7894	5.8502	5.9124	5.9758	6.0405	6.1066	6.1742	6.2432	Mean differences not sufficiently accurate				
81°	6.3138	6.3859	6.4596	6.5350	6.6122	6.6912	6.7720	6.8548	6.9395	7.0264					
82°	7.1154	7.2066	7.3002	7.3962	7.4947	7.5958	7.6996	7.8062	7.9158	8.0285					
83°	8.1443	8.2636	8.3863	8.5126	8.6427	8.7769	8.9152	9.0579	9.2052	9.3572					
84°	9.5144	9.6768	9.8448	10.019	10.199	10.385	10.579	10.780	10.988	11.205					
85°	11.430	11.664	11.909	12.163	12.429	12.706	12.996	13.300	13.617	13.951					
86°	14.301	14.669	15.056	15.464	15.895	16.350	16.832	17.343	17.886	18.464					
87°	19.081	19.740	20.446	21.205	22.022	22.904	23.859	24.898	26.031	27.271					
88°	28.636	30.145	31.821	33.694	35.801	38.188	40.917	44.066	47.740	52.081					
89°	57.290	63.657	71.615	81.847	95.489	114.59	143.24	190.98	286.48	572.96					

NO: 11179

Cephalopods: handling, processing and products

by
R. Kreuzer